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# An Analytical Study of Optimal Control within a Mathematical Model of Smoking Behavior

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**Abstract**

In this work a dynamical model of smoking is presented. There are so many harmful effects of smoking on our health. Smoking is a source of the air pollution and it puts the life of others at risk that lives in the nature. The model is categorized into five classes. These include non-smoking individuals, non-regular smokers, active smoking individuals, those people who have given up smoking temporarily and the people who quit the smoking on permanent basis. This mathematical smoking model depends on four optimal controls, i.e., education effort against smoking, against smoking gum, against an alkaloid poison that occur in tobacco and smoking banned by government in public areas, all are defined in the model. The solution to the optimal control problem is derived using the Pontryagin Maximum Principle (PMP) and an analytical approach has been adopted for this purpose.

**Keywords:** mathematical dynamic model; analytical solutions; Pontryagin maximum principle (PMP); optimal control.

**Introduction:**

According to the scientific research smoking is injurious for our health. In 2019 the report was publish by Global tobacco Epidemic, every year eight million people are died due to smoking and its major causes are lungs tumor, tuberculosis, certain eye diseases and problems of the immune system, smoking is major risk for rheumatoid arthritis cardiovascular, respiratory diseases and heart attack, amazing fact was reported in 2018, there are 1.377 billion peoples are active smokers in this world, including both male and female smokers and other forty three million children under 13 to 15 years of age are involved in smoking, as per data collection from World Health Organization (WHO) every year 10 million individuals are died due to smoking, 65000 children died every year due to breath because of air polluted by smokers [1,2]. Smoking is a bad habit and it has negative impact on ecological environment, It badly affects youngster life, smoking can change youngster personal sexual habits, social activity and other major factors, smoking can cause infection diseases in our body [3]. Brownlee was mathematician, who proposed a diseases model into mathematics with the help of probability he took three years and gave us law about the spread of infection. Ker mark and Mc Kendrick was other mathematicians who introduced a model in which they analyzed infection which was caused by smoking. Smoking is same as the other infectious diseases are existed; smoking is common diseases spread in the entire world. In 16$^{th}$ century, Columbus was introduced with smoking in Europe. Cigarette was introduced in last 19$^{th}$ century. The first cigarette manufacturer speed was 200 units per minute and nowadays it is 9000 [4,5]. One cigarette contains approximately 4000 chemical compound, at least 200 chemicals are poisons for human health, while another 44 chemicals are harmful and can affect human in tumor diseases, a cigarette smoke is dangerous and it can be a source of different diseases like tumor, depression, hepatitis C, heart diseases, liver diseases, lungs diseases etc. Cigarettes can badly affect the health of both men and women equally and specially for under 18 year boys or girls. The major ingredient in the cigarette is the nicotine. It has tendency to affect human brain system and nervous system of the human body. Smoking produces Tar in the body. This Tar causes many problems such as urine problem, blood pressure, and human growth problem. Hydrogen cyanide is found in cigarette it has a tendency to affect human lungs, loss of strength, and loss of energy of brain, it causes pain in human brain, and it makes you sick. Scientific studies have found that the major cause of the reduction in white blood cells and plasma blood cells is the cigarette inhaling and as a result of this inhaling there appears risk of blood tumor. Cigarette contains arsenic oxide. High level of smoking produces high level of arsenic oxide which becomes the source of urinary tract tumor, kidney tract tumor, lung tumor, liver tumor, and skin tumor. Cadmium found in cigarette smoke. When smoker inhale cigarette smoke into his or her body high level of cadmium causes sensory disorders, kidney failure, diarrhea, vomiting, muscle cramps, seizures, and throat tumor.

Mathematical modelling is considered to be a very effective tool to understand the world problems and their dynamical behaviors. In Mathematical modelling we become able to give a mathematical formulation of the real world problem. Extensive research is done on smoking by making use of mathematical approaches [1-6]. Many mathematical models are developed on the proposed assumptions by the researchers. These results can be used to devise efficient strategies to curb the smoking.

The model is split into four different subcategories in [7,9]. The initial group comprises prospective smokers who have not yet begun smoking. The second group consists of people who only smoke 20 cigarettes per day or less. The third group is made up of people who smoke more than 20 cigarettes daily; the fourth category consists of those who have successfully stopped smoking. Two control variables are included in the model to affect smoking habit: the first symbolizes antismoking campaigns and the second reflects media-based awareness activities.

Building on the framework originally developed in [6,10], researched a smoking behaviour model. In [9,11,12], the population is separated into five different subgroups: potential smokers, occasional smokers, active smokers, people who have temporarily stopped smoking, and those who have permanently quit. In [5,13], a comparable five-category categorization is offered. The mathematical model presented in [5,14] is taken in the current study. This study primarily aims to present four control methods designed to reduce the number of smokers and increase the number of non-smokers.

Outline of the articles are given in following way. Firstly mathematical smoking model is discussed. After then formulation of the dynamical model is given with the help of optimal control for finding solutions, results and discussion. In the end conclusion is given.

**Dynamic Model:**

Consider the dynamic mathematical model where total population $N(x)$ with respect to time, is divided into five subdivision groups, i.e., $O(x)$ is the population or some time smoking, $P(x)$ is the population of not smoking, $Q(x)$ is representing smokers who possess energetic approach to smoking, $R_t(x)$ shows the peoples that have quit the smoking habit temporarily whereas $R_p(x)$ is representing the peoples who have quit smoking habit forever. The above study dynamic model [7] is defined as follows:

$$\frac{dO}{dx} = \gamma - \alpha O(x)Q(x) - \sigma O(x)$$

$$\frac{dP}{dx} = \alpha O(x)Q(x) - \beta_1 P(x) - \sigma P(x)$$

$$\frac{dQ}{dx} = \beta_1 P(x) + \beta_2 Q(x)R_t(x) - (\sigma + \lambda)Q(x) \qquad \text{(i)}$$

$$\frac{dR_t}{dx} = -\beta_2 Q(x)R_t(x) - \sigma R_t(x) + \lambda(1 - \mu)Q(x)$$

$$\frac{dR_p}{dx} = \mu\lambda Q(x) - \sigma R_p(x)$$

Furthermore, this diagram represented mathematical model of the below figure 1.

Total population $N(x)$

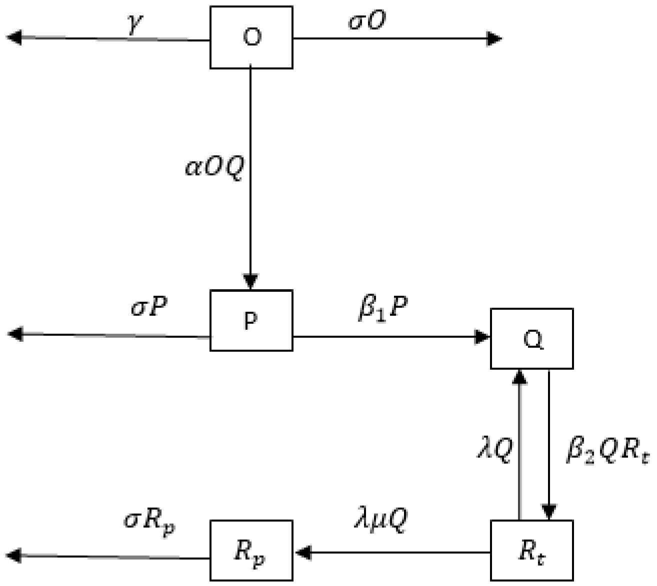


**Figure 1:** Compartment diagram

These smoking dynamic models depend on four optimal controls, i.e., education effort against smoking, against smoking gum, against an alkaloid poison that occur in tobacco and smoking banned by government in public areas; all are defined in the model. Our aims are to find the number of reduce smoking people and find the number of increasing smoker who have stop smoking we are using here minimize the cost function.

Description of the parameters used in the dynamic model is as under, $\gamma$ is natural death rate of for some time smoking $O(x)$, $\sigma$ is the natural death rate, $\alpha$ is the efficient contact rate between smoker $Q(x)$ and for some time smoking $O(x)$, $\beta_1$ is the transition rate from occasional smoker $P(x)$ become regular smoker, $\beta_2$ is representing a relationship between the people who do smoking and those who quit smoking but resumed it again, $\lambda$ is the rate of people who stop smoking, $\mu$is the rate of people who stop smoking permanently $R_p(x)$.

**Optimal Control:**

**(i) Optimal Control Formulations**

Emerging in the 1950s, the idea of optimum control offered a methodical method for finding best strategies within dynamic systems. Two main approaches have emerged to address optimal control issues: dynamic programming, as suggested in [10,14,15], and Pontryagin's Maximum Principle (PMP), first introduced in [11,13]. The main goal of optimum control is to find the most efficient input functions satisfying given physical constraints, optimizing a given performance criterion, therefore maximizing benefits or minimizing expenses[16,17].

Our aim is to satisfy minimizing or maximizing this optimal control constraint. The initial optimal control start from at time $x_0, t(x_0)$, to the final optimal control time $x_f, t(x_f)$ in this way the result defined as objective function will be maximum or minimum value.

In this work, four controls are considered. Eq. (1) represents the controls, when merging them into standard model it will take following form:

$$\frac{dO}{dx} = \gamma - \alpha O(x)Q(x) - (\mu + v_1 + v_4)\, O(x)$$

$$\frac{dP}{dx} = \alpha O(x)Q(x) - (\beta_1 + \sigma + v_2 + v_4)P(x)$$

$$\frac{dQ}{dx} = \beta_1 P(x) + \beta_2 Q(x)R_t(x) - (\sigma + \lambda + v_3 + v_4)Q(x) \qquad (2)$$

$$\frac{dR_t}{dx} = -\beta_2 Q(x)R_t(x) - (\sigma + v_2 + v_4)R_t(x) + \lambda(1 - \mu)Q(x)$$

$$\frac{dQ_p}{dx} = \mu\lambda Q(x) - \sigma R_p(x) + (v_1 + v_4)O(x) + (v_2 + v_4)P(x) + (v_3 + v_4)Q(x) + (v_2 + v_4)R_t(x)$$

Controls provide a pivotal role in reducing the number of the smokers and make the potential smokers to a lower level. It is vital to put four controls. If these four controls were not put then there would have been increase in the individual smokers and the potential smokers. In the establishment of the objective function, control problems given in equation (2) were considered. The objective function obtained is given below.

$$J(u(x)) = \int_0^x f(Q(x) - R_p(x) + \frac{1}{2}(k_1 {v_1}^2(x) + k_2 {v_2}^2(x) + k_3 {v_3}^2(x) + k_4 {v_4}^2(x))dx\ldots\ldots \qquad (3)$$

The main goal of this study is to minimize $J(u(x))$ subject to its constraints by utilizing the optimal control method.

**(ii) Optimal Control Existence**

Suppose we write Equation (2) as follows:

$$\varphi(x) = C(\varphi) + F(\varphi) \qquad (4)$$

Matrix form of this equation written as,

$$\varphi = [O(x), P(x), Q(x), R_t(x), R_p(x)]^t,$$

And

$$C = \begin{bmatrix} -a & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 \\ 0 & \alpha_1 & -c & 0 & 0 \\ 0 & 0 & d & -e & 0 \\ f & g & h & m & -\sigma \end{bmatrix}$$

where,

$a = (\sigma + v_1 + v + 4)$

$b = (\beta_1 + \sigma + v_2 + v_4)$

$c = (\sigma\lambda + v_3 + v_4)$

$d = \lambda(1 - \mu)$

$e = (\sigma + v_2 + v_4)$

$f = v_1 + v_4$

$g = v_2 + v_4$

$h = \mu\lambda + v_3 + v_4$

$i = v_2 + v_4$

$F(\varphi) = (\gamma - \alpha OQ\ \alpha OQ\ \beta_2 QR_t - \beta_2 QR_t\ P\ )$

A nonlinear differential equation with bounded coefficients shows in Eq. (4) is, where derivative of $\varphi(x)$ is with respect to time derivative of $\varphi$.

We have:

$$A(\varphi) = C(\varphi) + F(\varphi)$$

$$F(\varphi_1) - F(\varphi_2) = \big(-\alpha O_1 Q_1 + \alpha O_2 Q_2 + \alpha O_3 Q_3 + \alpha O_4 Q_4\ \alpha O_1 Q_1 - \alpha O_2 Q_2 - \alpha O_3 Q_3 - \alpha O_4 Q_4\ \beta_2 Q_1 R_{t_1} - \beta_2 Q_2 R_{t_2} - \beta_2 Q_3 R_{t_3} - \beta_2 Q_4 R_{t_4} - \beta_2 Q_1 R_{t_1} + \beta_2 Q_2 R_{t_2} + \beta_2 Q_3 R_{t_3} + \beta_2 Q_4 R_{t_4}\ P\ \big)$$

Therefore,

$$|F(\varphi_1) - F(\varphi_2)| = |-\alpha O_1 Q_1 + \alpha O_2 Q_2| + |\alpha O_1 Q_1 - \alpha O_2 O_2| + \left|\beta_2 Q_1 R_{t_1} - \beta_2 Q_2 R_{t_2}\right| + \left|-\beta_2 Q_1 R_{t_1} + \beta_2 Q_2 R_{t_2}\right|$$

$$\leq 2\alpha|O_1 Q_1 + O_2 Q_2| + 2\beta_2\left|Q_1 R_{t_1} + Q_2 R_{t_2}\right|$$

$$= 2\alpha|Q_1(O_1 - O_2) + O_2(Q_1 - Q_2)| + 2\beta_2|R_{t_1}(Q_1 - Q_2) + Q_2(R_{t_1} - R_{t_2}| \leq (2\alpha|O_2| + 2\beta_2|R_{t_1}|)|Q_1 - Q_2| + 2\alpha|Q_1||O_1 - O_2| + 2\beta_2|Q_2||R_{t_1} - R_{t_2}|$$

$$\leq (2\alpha + 2\beta_2)\frac{\gamma}{\sigma}|Q_1 - Q_2| + 2\alpha\frac{\gamma}{\sigma}|O_1 - O_2| + 2\beta_2\frac{\gamma}{\sigma}|R_{t_1} - R_{t_2}|$$

Given statement $|A(\varphi_1) - A(\varphi_2) \leq Z|\varphi_1 - \varphi_2|$,
where $Z = max\{(2\alpha + 2\beta_2)\gamma/\ \sigma, ||C||\} < \infty$. It is evident that $A(\varphi)$ is uniformly Lipschitz continuous, it is revealed from the definition of $u_i$ that there is existence of the solution of the controlled system (4.1).

**Optimal Control Solutions**

Suppose that the Hamiltonian function written as follows in optimal control Equations. (2)-(3).

$$H = L + \Sigma_{j=1}{}^{5} \varrho_j(t) g_j \tag{5}$$

Where $L$ is known as the Lagrangian function and can be given as follows:

$$L(Q, R_t, v_i) = A_1 Q(t) - A_2 R_p(t) + \frac{1}{2}\left[k_1 v_1{}^2(t) + k_2 v_2{}^2(t) + k_3 v_3{}^2(t) + k_4 v_4{}^2(t)\right] \tag{6}$$

Where,

$$g_1 = \frac{dO}{dx}, g_2 = \frac{dP}{dx}, g_3 = \frac{dQ}{dx}, g_4 = \frac{dR_t}{dx}, g_5 = \frac{dR_p}{dx} \tag{7}$$

Hamiltonian function is obtained as follows:

$$\begin{aligned} H = (Q(x) - R_p(x) + \frac{1}{2}(k_1 v_1{}^2(x) + k_2 v_2{}^2(x) + k_3 v_3{}^2(x) + k_4 v_4{}^2(x)) + \varrho_1(\gamma \\ - \alpha O(x)Q(x) - \sigma O(x)) + \varrho_2(\alpha O(x)Q(x) - \beta_1 P(x) - \sigma P(x)) \\ + \varrho_3(\beta_1 P(x) + \beta_2 S(x)R_t(x) - (\sigma + \lambda)Q(x)) + \varrho_4(-\beta_2 Q(x)R_t(x) \\ - \sigma R_t(x) + \lambda(1 - \mu)Q(x) + \varrho_5 \mu\lambda Q(x) - \sigma R_p(x)) \end{aligned}$$

Adjoint variables obtained by using PMP method.

## Results and Discussions

We are using dynamic model for solving solution of optimal control problem. $\lambda$, $\mu$, $\beta$, $\alpha_1$, $\alpha_2$, $\gamma$ and $\sigma$ these are all controlling parameter of dynamic model, if we put different values in different parameters it represents the population of smokers increase or decrease in the graph of the optimal control tobacco of smoking model. Optimal Control Existence we used matrix and non linear ordinary differential equation for unbounded coefficients, For Optimal Control Solutions, we used Hamiltonian function and Lagrangian function.

Solutions of the optimal control problem is given by utilizing PMP method where Runge kutta method up to fourth order employed. The dynamic behavior of the smoking model in the absence of control interventions revealed notable trends. The population of possible smokers first decreased dramatically, but over time, this was followed by a modest rise. The number of occasional smokers also showed an upward trend; the active smoker population, however, clearly declined. The number of smokers who temporarily stopped smoking was increased. The number of persons who stopped smoking was increased.

**Conclusion:**

In this paper, solution of the optimal control tobacco smoking using dynamic model, dynamic model is using for flow of smoking, we divide mathematical model into five subclasses and dynamic model depend on four optimal controls, shows in education effort against smoking $v_1$, against smoking gum $v_2$, against nicotine drug $v_3$ and smoking banned by government in public areas $v_4$, result of this paper certain analytical the aim of solving this model, increase or decrease the population of smokers who are smoking temporary and permanently shows in optimal control formulation, Hamiltonian function and Lagrangian function is use in PMP method for proving optimal control solution of tobacco smoking model. Our aim was solving smoking model analytically.

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