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Efficient Explicit Improved Scheme for Numerical Solution of Cauchy Problems

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Abstract:

This study work is focus on to design explicit splendid converging method to estimate numerical result of initial-value-problems (IVP's) for differential equation in nature ordinary. The stability criteria of the improved scheme are investigated, and corresponding stability region is depicted. Error analysis carried out also affirms accuracy having third order. Error (LTE) is derived by utilizing Taylor's series expansion to skip higher order term after matching corresponding coefficients. Adding a partial derivative inside function evaluation raises convergence and reduces both errors (maximum and last). Based on results for improved scheme to illustrate the accuracy and efficiency. The comparisons of the improved and existing schemes having same order of local accuracy are discussed. Looking at numerical results, the improved scheme gives the better result in the comparison of existing few same order method. Stability is proved to examine behavior of method and its region is visualized. Consistency is investigated that is showing how error shrink to zero. the numerical result is validated and illustrated graphically via utilizing MATLAB 2023a software.

Keywords: *Numerical Methods, Third order, Local Truncation Error, Convergence*

I. INTRODUCTION

Ordinary differential equations (ODEs) play a crucial role in modelling problems arising in science and engineering. In particular, Cauchy problems (or initial value problems - IVPs) are encountered in population dynamics, fluid dynamics and control theory. These problems may not always have analytical solutions, and there is a need for efficient and accurate numerical techniques. Many numerical methods have been proposed to find approximate solutions with different levels of accuracy. But there is a need for fast and accurate schemes. In this work, our goal is to address this issue by designing a new numerical integration technique. The goal is to improve the accuracy of the method without compromising its simplicity.

We have developed an efficient explicit third-order improved method for solving Cauchy problems for ordinary differential equations. The modification of the new scheme is based on the adjustment of classical numerical schemes and introducing correction terms. They are done to improve the order of convergence of the scheme,

while keeping it efficient. This scheme is explicit, easy to implement and efficient. It does not require the solution of nonlinear equations, as is required by implicit schemes. The scheme is based on the method, which has third-order accuracy. Hence, it can be used to solve many problems.
The proposed scheme is derived using a Taylor series expansion, a standard approach in numerical analysis to develop higher-order methods. The series expansion of the exact solution is used to compare the coefficients with the numerical solution, thus revealing the local truncation error. Terms of higher order are discarded to obtain the desired accuracy. The presence of partial derivative terms in the evaluation of the function increases the accuracy. This facilitates the tracing of solution curve in each step. This leads to better convergence of the proposed scheme. The scheme is theoretically proven to be third-order accurate.

A detailed error analysis for the scheme is conducted. The local truncation error (LTE) is computed by Taylor series expansion. The analysis confirms the order of the leading error term as four and thus the convergence of the scheme as third order. Besides LTE, the global error is also investigated by numerical experiments. These results indicate the maximum error as well as the final error is significantly reduced. Hence, the new scheme is superior to conventional approaches. This shows the advantage of adding more correction procedures.

Another important aspect of the method is the stability. We investigate the stability of the method to assess its performance under different step sizes. We use a suitable test equation to get the stability function. Using this, we compute and draw the stability region. The plot gives an idea of how the method will perform in various cases. We observe that the new scheme has a good stability region. As a result, the scheme can be applied to solve a wide variety of problems without stability problems.

Theoretical analysis also indicates that the new scheme is consistent. A numerical scheme is said to be consistent if the local truncation error (LTE) approaches zero as the step size shrinks to zero. The LTE formula derived is indicative of this. This guarantees the numerical solution will converge to the analytic solution as the step size goes to zero. This, together with stability, ensures convergence as per the theory of numerical analysis. Hence, the scheme meets the necessary conditions for a good numerical scheme. This enhances the proposed method's reliability.

To assess the effectiveness of the new approach, a number of examples are presented. These are chosen to demonstrate the accuracy and efficiency of the scheme. These are compared with the solutions of existing third-order schemes. Results are presented in tables and figures to clearly show the differences. The enhanced method has better accuracy with smaller error measures. This proves it to be better than traditional schemes of the same level. The results demonstrate the efficiency and accuracy of the proposed scheme.

All numerical simulations and graphical validations are performed in MATLAB 2023a. The proposed method is easy to apply as it is explicit. Plots of the exact and approximate solutions visually verify the accuracy of the scheme. The results demonstrate that the new scheme is close to the exact solution. This confirms the theoretical results obtained in the paper. In conclusion, the proposed method is accurate, stable and efficient. This method can thus be seen as an important addition to the numerical solution of ordinary differential equations.

2. DERIVATION

We initiate by considering general form of ODE's

$$\frac{dy}{dx} = f(x_n, y_n), \quad y(x_0) = y_0 \qquad (1)$$

It is assumed, eqn. (I) on a specific integration interval confesses a unique solution. Exact and numerical result is signified by $y(x)$ and y_n respectively. Integare eqn. (I), it adopts new form

$$y_{n+1} = y_n + \int_{x_0}^{x_0+h} f(x_n, y_n)\, dx$$

In discrete manner

$$y_{n+1} = y_n + h\sum_{i=1}^{s} \rho_i w_i$$

If $i = 1,2,3$

$$y_{n+1} = y_n + h(\rho_1 w_1 + \rho_2 w_2 + \rho_3 w_3) \qquad (2)$$

Where w_1, w_2 and w_3 are the slopes, determined by

$$w_1 = f(x_n, y_n)$$

$$w_2 = f\left(x_n + \alpha_2 h, y_n + h(\beta_{21} w_1) + h^2(\delta_{21} w_1) f_y\right)$$

$$w_3 = f\left(x_n + \alpha_3 h, y_n + h(\beta_{31} w_1 + \beta_{32} w_2) + h^2(\delta_{31} w_1) f_y\right)$$

Taylor Series (TS) expension for $Y(x_n, y_n)$ is

$$G(x_n, y_n) = y(x) + hf + h^2\left[\frac{1}{2} f_x + \frac{1}{2} f_y f\right] + h^3\left[\frac{1}{6} f_{xx} + \frac{1}{3}\left(f_{xy} + \frac{1}{2} f_{yy} f + \frac{1}{2} f_y^2\right) f + \frac{1}{6} f_x f_y\right]$$
$$+h^4\left[\begin{array}{l}\frac{1}{24} f_{xxx} + \frac{1}{24}(3 f_{xxy} + 3 f_{xyy} f + 5 f_y f_{xy}) f + \frac{1}{8} f_x f_{xy} + \frac{1}{24} f_{yyy} f^3 + \\ \frac{1}{24}(3 f_y f^2 f_{yy} + 3 f_x f f_{yy} + f_y^3) f + \frac{1}{24}(f_x f_y^2 + f_y f_{xx})\end{array}\right] + O\left(h^5\right) \quad (3)$$

Slopes w_2 and w_3 are expended thru using Taylor's Series then surrogate w_1, w_2 and w_3 into (2). Finally coefficients of are equating with (3) in power of h up to h^3 for attaining system.

$$\rho_1 + \rho_2 + \rho_3 = 1 \qquad \alpha_2 \rho_3 \beta_{32} = \frac{1}{6} \qquad \alpha_2 \rho_2 + \alpha_3 \rho_3 = \frac{1}{2}$$
$$\frac{1}{2}(\alpha_2^2 \rho_2 + \alpha_3^2 \rho_3) = \frac{1}{6} \qquad \rho_2 \beta_{21} + \rho_3 \beta_{31} + \rho_3 \beta_{32} = \frac{1}{2} \qquad \alpha_2 \rho_2 \beta_{21} + \alpha_3 \rho_3 \beta_{31} + \alpha_3 \rho_3 \beta_{32} = \frac{1}{3} \quad (4)$$
$$\rho_2 \delta_{21} + \rho_3 \delta_{31} + \rho_3 \beta_{21} \beta_{32} = \frac{1}{6} \qquad \frac{1}{2}(\rho_2 \beta_{21}^2 + \rho_3 \beta_{31}^2 + \rho_3 \beta_{32}^2) + \rho_3 \beta_{31} \beta_{32} = \frac{1}{6}$$

Above sysyem is nonolinear which has equationas and 10 variable quantities. Now we investigate this sytem for solution. So it has multiple solution because of free variables. One of the accurate amnd reliably solution we have

$$w_1 = f(x_n, y_n)$$
$$w_2 = f\left(x_n + \frac{2h}{3}, y_n + hw_1\left(\frac{2}{3} + hf_y\right)\right)$$
$$w_3 = f\left(x_n + \frac{2h}{3}, y_n + \frac{1}{15} hw_1 + \frac{3}{5} hw_2 - \frac{4}{5} h^2 w_1 f_y\right) \quad (5)$$
$$y_{n+1} = y_n + \frac{h}{12}(3w_1 + 4w_2 + 5w_3)$$

New proposed method has deriven now further investigation is mandatory to check accuracy , convergence and compute error.

3. LOCAL TRUNCATION ERROR

The enhanced extended method having local truncation error is expressed as T_{n+1} where

$$T_{n+1} = G(x+h) - y_{n+1}$$

Where $G(x+h)$is gained by TS and $y_{n+1}$is used as an approximate solution. Taylor series is utilized to expand these around x and similar terms collect in h. Now we expand proposed accelerated explicit Method present in eqn (2) upto h^4, we get

$$T_{n+1} = \begin{pmatrix} (\frac{1}{216} f_{yyy} f + \frac{49}{648} f_{xyy}) f^2 - \frac{1}{72} f_{xy} (ff_y - f_x) - \frac{1}{72} f_y f_{xx} + \frac{49}{648} f_{xxy} f \\ + \frac{19}{24} f_y^3 f - f_{yy} (\frac{5}{6} f_y f^2 - \frac{1}{72} f_x f) - \frac{5}{81} f_{xyy} f + \frac{1}{24} f_x f_y^2 + \frac{1}{216} f_{xxx} \end{pmatrix} h^4 + O(h^5) \tag{6}$$

4. CONSISTENCY ANALYSIS

Definition 4.1 NM together IVP with an increment function $\eta(x_n, y_n; h)$ is accepted to be consistent,if

$$\lim_{h \to 0} \eta(x_n, y_n; h) = f(x_n, y_n)$$

From proposed method

$$\eta(x_n, y_n; h) = \frac{1}{12}(3w_1 + 4w_2 + 5w_3)$$

Proceeds $\lim_{h \to 0}$ on both side

$$\lim_{h \to 0} \eta(x_n, y_n; h) = \frac{1}{12} \lim_{h \to 0} (3w_1 + 4w_2 + 5w_3)$$

$$= \frac{1}{12} \lim_{h \to 0} \begin{pmatrix} 3f(x_n, y_n) + 4f\left(x_n + \frac{2h}{3}, y_n + hw_1\left(\frac{2}{3} + hf_y\right)\right) + \\ 5f\left(x_n + \frac{2h}{3}, y_n + \frac{1}{15}hw_1 + \frac{3}{5}hw_2 - \frac{4}{5}h^2 w_1 f_y\right) \end{pmatrix}$$

$$= f(x_n, y_n)$$

Hence proved the consistent with at least third order accuracy for improved method.

5. LINEAR STABILITY ANALYSIS

Stability is shown by taking Dahlquist's test problem which is in form

$$\frac{dy}{dx} = q\,y(x); \quad y(0) = y_0, \quad q \in C$$

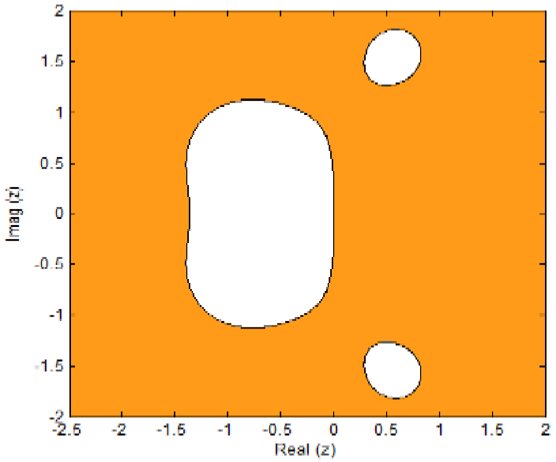
we have attained polynomial function,we called it stability polynomial by emplying (5) on this test problem. In figure I unshaded region dosplays linear stability region.

$$w_1 = q\,y_n;\; w_2 = q\,y_n\left[1 + \frac{2}{3}hq + h^2q^2\right];\; w_3 = q\,y_n\left[1 + \frac{2}{3}hq - 6h^2q^2 + 3h^3q^3\right]$$

Substituting above all values in (5) ,the stability function is originated

$$R(z) = 1 + z + \frac{z^2}{2}\left(1 + \frac{z}{3} + \frac{z^2}{2}\right) \text{ where } z = hq$$

Figure 1. The 2D Stability region for the Proposed Method



6. NUMERICAL INVESTIGATION

To investigate the behaviour of methods either they have better converges or not, we examined and tested few problems for proving better result. We have choosen three methods, two method are taken from open literaute and one is proposed. Before taking method, one thing is keep in mind all methods have same order. So here with the help of MATLAB2023a, error such absolute last and maximum is evaluated by using these methods, graph of each problem is displayed separately. In graph three curvy lines are drawn . Numericalll and graphically result give the clear evidence on which proposed method is better in all aspects. RK3M and RK3HM both less converges than developed one.

Table 1. Shows Numerically Illustration of Problem 1

Problem 1. $\frac{dy}{dx} = xy^3 - y, \quad y(0) = 1$, $Exact = \frac{2}{\sqrt{2+4x+2e^{2x}}}$			
Step-size /method	RK3HM	RK3M	Proposed
0.1	2.2406e-004	2.1008e-005	5.8246e-006
	2.1363e-004	1.5944e-005	2.2251e-006
	0e0	0e0	0e0
0.05	5.1698e-005	2.5084e-006	5.7261e-007
	4.9489e-005	1.8970e-006	6.9545e-008
	0e0	0e0	0e0
0.025	1.2497e-005	3.0722e-007	6.2151e-008
	1.1988e-005	2.3198e-007	3.1638e-009
	0e0	0e0	0e0
0.0125	3.0773e-006	3.8027e-008	7.1587e-009
	2.9553e-006	2.8697e-008	1.1110e-009
	0e0	0e0	0e0

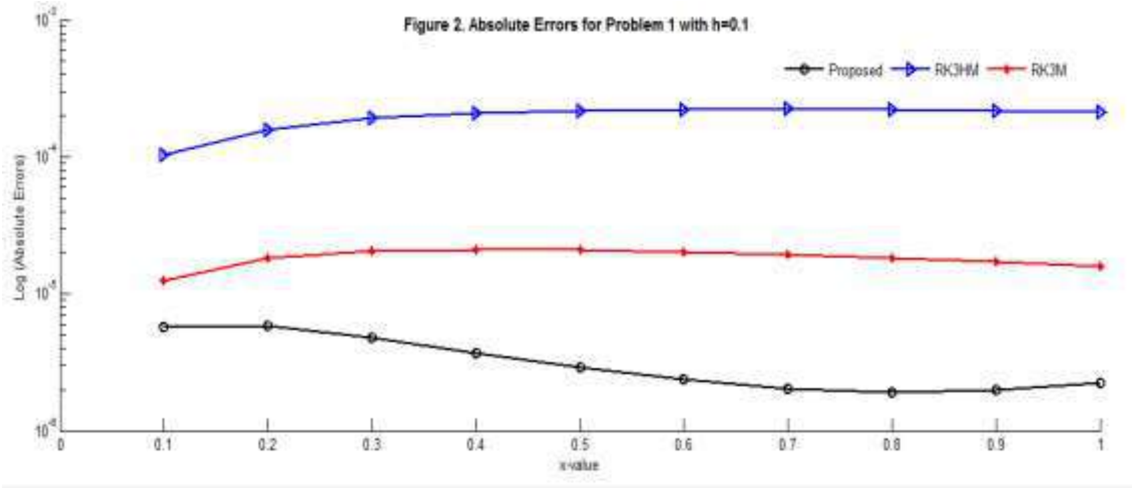


Figure 2. Absolute Errors for Problem 1 with h=0.1

Table 2. Shows Numerically Illustration of ProblemI

Problem 2 , $Exact = \frac{1}{3\sqrt{9+6x^3}}$ $\frac{dy}{dx} = \frac{x^2}{y}$, $y(0)=1$			
Step-size /method	RK3HM	RK3M	Proposed
0.1	1.5241e-003	2.2290e-005	2.6463e-006
	1.5241e-003	2.2290e-005	2.0957e-006
	0e0	0e0	0e0
0.05	3.8551e-004	2.8590e-006	4.1078e-007
	3.8551e-	2.8590e-	4.1078e-007

	004	006	
	0e0	0e0	0e0
0.025	9.7000e-005	3.6139e-007	6.1315e-008
	9.7000e-005	3.6139e-007	6.1315e-008
	0e0	0e0	0e0
0.0125	2.4332e-005	4.5408e-008	8.3075e-009
	2.4332e-005	4.5408e-008	8.3075e-009
	0e0	0e0	0e0

Figure 3. Absolute Errors for Problem 2 with h=0.1

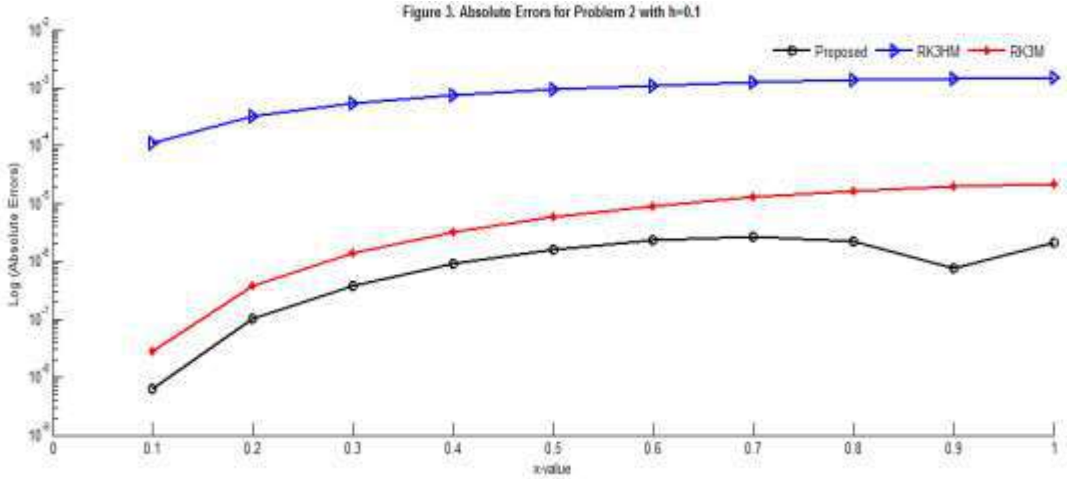


Table 3. Shows Numerically Illustration of Problem 3

Problem 3. $Exact=\sqrt{3-2\cos x}\ \frac{dy}{dx}=\frac{\sin x}{y},\ y(0)=1$			
Step-size /method	RK3HM	RK3M	Proposed
0.1	2.2846e-003	1.4261e-005	5.1507e-006
	1.9600e-003	1.3062e-005	3.8481e-007
	0e0	0e0	0e0
0.05	6.5681e-004	1.8175e-006	6.2727e-007
	5.5494e-004	1.6664e-006	3.3522e-008
	0e0	0e0	0e0
0.025	1.8632e-004	2.2894e-007	7.7070e-008
	1.5560e-004	2.0996e-007	9.3205e-009
	0e0	0e0	0e0
0.0125	5.2173e-005	2.8709e-008	9.5465e-009
	4.3185e-005	2.6336e-008	1.4863e-009
	0e0	0e0	0e0

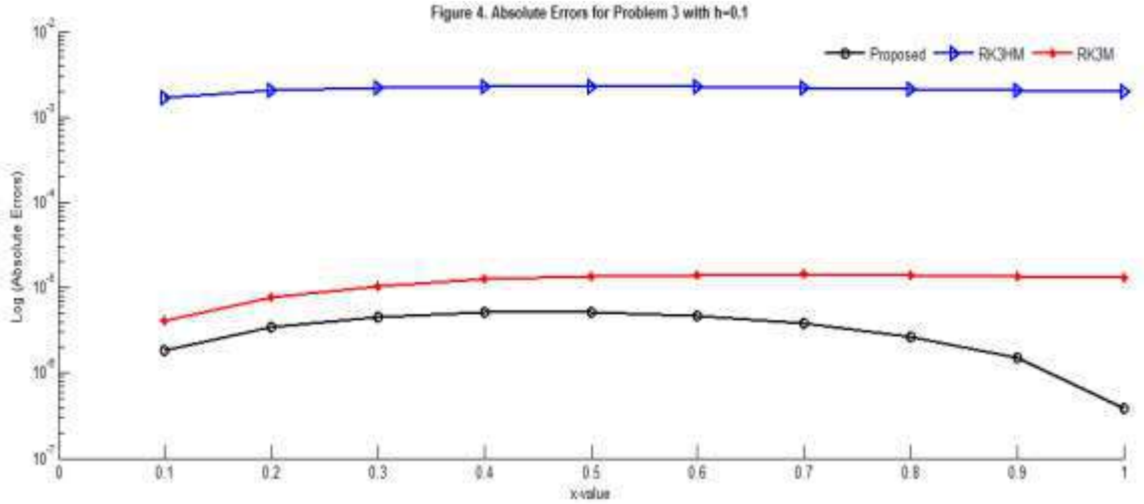


Figure 4. Absolute Errors for Problem 3 with h=0.1

Table 4. Shows Numerically Illustration of Problem 4

			Problem 4.
,	$Exact = e^{\frac{x^3}{3}}$	$\frac{dy}{dx} = x^2 y,$	$y(0)=1$
Step-size /method	RK3HM	RK3M	Proposed
0.1	3.1171e-003	6.3568e-005	1.4165e-005
	3.1171e-003	6.3568e-005	1.3359e-005
	0e0	0e0	0e0
0.05	8.2411e-004	8.4079e-006	1.6414e-006
	8.2411e-004	8.4079e-006	1.2474e-006
	0e0	0e0	0e0

0.025	2.1194e-004	1.0805e-006	1.9677e-007
	2.1194e-004	1.0805e-006	1.2857e-007
	0e0	0e0	0e0
0.0125	5.3744e-005	1.3694e-007	2.4068e-008
	5.3744e-005	1.3694e-007	1.4347e-008
	0e0	0e0	0e0

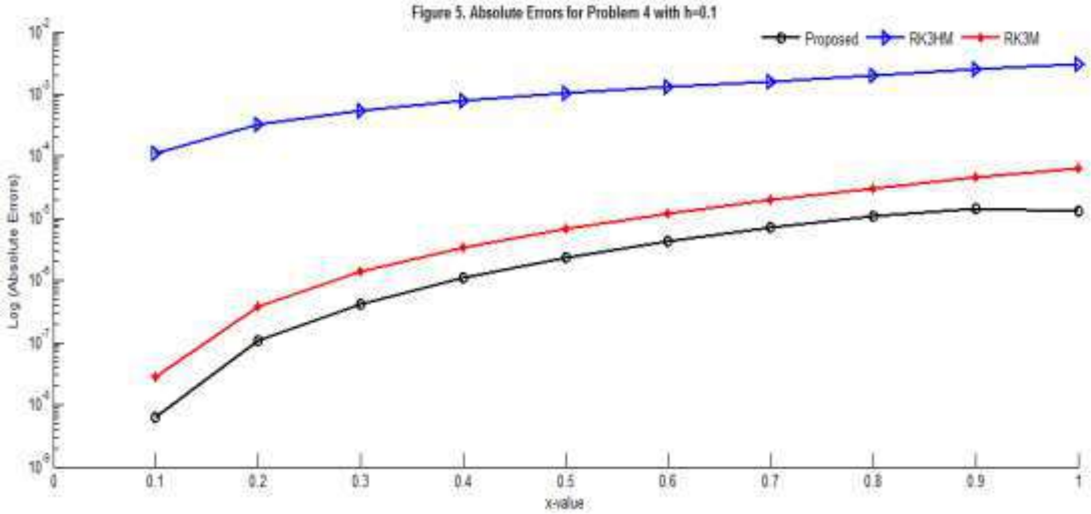


Figure 5. Absolute Errors for Problem 4 with h=0.1

7. RESULTS AND DISCUSSIONS

The proposed method is constructed to find more suitable and accurate solution of IVP's for ODE's. Proposed method has accuracy order three so we compare it with order third methods. AT different step size such as 0.1, 0.05, 0.025 and 0.0125 error is computed including absolute last and maximum and CPU time is observed in seconds. From the data and graphs listed above clearly observed that the extended proposed scheme produced a smaller error then existing method, having the same order of accuracy and CPU time. The results have achieved numerically by proposed method which comes quickly closer to exact point of the solution in comparison of RK3M and

RK3HM. Conclusively, the amended proposed scheme is most effective to solve Cauchy differential equations and converges faster to the required solution.

7. CONCLUSION

This study work had achieved these objectyive, firstly it focus on to design explicit splendid converging method to estimate numerical result of IVP's for differential equation in nature ordinary. Secondly, The stability criteria of the improved scheme are investigated, and corresponding stability region is depicted. Thirdly, Error analysis carried out also affirms accuracy having third order. Error (LTE) is derived by utilizing Taylor's series expension to skip higher order term after matching corresponding coefficients. Adding a partial derivative inside function evalution raises convergence and reduces both errors (maximum and last). Based on results for improved scheme to illustrate the accuracy and efficiency. The comparisons of the improved and existing schemes having same order of local accuracy are discussed. Looking at numerical results, the improved scheme gives the better result in the comparison of existing few same order method. Stability is proved to examine behavior of method and its region is visualized. Finally, Consistency is investigated that is showing how error shrink to zero. the numerical result is validated and illustreated graphically via utilizing MATLAB 2023a software

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