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A Theoretical Investigation of Local Truncation Error and Convergence of an Accelerated Third-Order Runge–Kutta Method for Non-Autonomous Ordinary Differential Equations

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Abstract:

In this article, we theoretically examined well-organized third order numerical technique for IVP of ODE's including partial derivative which has enhanced its competency regarding truncation error. Accelerated numerical method for local truncation error and convergence is theoretically investigated to assess how accurate and reliable proposed method is. Expansion of Taylor's series is done that provides right pattern to expand and evaluate function evaluation. With the help of Taylor's series, Local truncation error is an explicitly derived to clarify the order of accuracy. Linear standard test is discussed for calculating stability. Stability is investigated for knowing behavior of method. Stability region is drawn by MATLAB2023a software. Stability region is visualized to check numerical method possess a bounded solution when applied frequently and repeatedly. Consistency is proved which tells us error goes to zero as much as we decrease step size which guarantees the desired result. Convergence criteria theoretically discussed here by proving consistency and stability. Theoretically findings showing that the numerical method is stable, attains high accuracy and gives reliable performance. Therefore, method is efficient and applicable for extensive class of IVP arising in the area of ODE's.

Keywords: Numerical Method, Local Truncation Error, Stability, Consistency, Convergence

I. INTRODUCTION

Ordinary differential equations (ODEs) take very crucial role in the formation of modeling arising in science, engineering, physics, biology, and economics. Many real-world problems are governed by initial value problems whose analytical solutions are either difficult or impossible to obtain in closed form. As a result, numerical methods have become indispensable tools for approximating solutions of such equations. When designing numerical schemes, computational efficiency is not the only consideration for obtaining reliable results; it's also important to understand the theoretical aspects of the schemes, such as their accuracy and stability.

One of the key elements in the analysis of numerical schemes is error analysis. The analysis of error helps determine the impact of discretization on the numerical solution and enables the comparison between different numerical schemes. In these studies, it is common to refer to local and global errors.

The local truncation error (LTE) is defined as the error presented in a single step of the numerical method, assuming that the earlier steps are exact. It serves as a primary measure for determining the order of accuracy. Specifically, if the LTE behaves like $O(h^{p+1})$, then the method has order p. Thus, the derivation and analysis of the LTE are essential for constructing of high-accuracy numerical methods.

Closely linked to error analysis is the concept of consistency, which guarantees that the consequence approximate technique precisely exemplifies the underlying differential equation as step size nearer to zero. Consistency guarantees that the discrete scheme is a faithful approximation of the continuous problem, forming a necessary condition for convergence. However, consistency alone is not sufficient to ensure reliable numerical results.

Another important property is stability, relating to the behavior of numerical errors in the iterative process. Stability is a property that guarantees that errors (due to truncation, round-off or small changes in initial conditions) do not blow up during the iterative process. This is crucial for stiff problems and when integrating over long times. A very popular method for investigating stability involves application of a standard linear test equation. Using such an approach, one establishes the so-called stability function, which describes the growth in error in each step.

The stability region is introduced as a part of this process. The region in the complex plane for which the numerical solution is bounded is known as the stability region. The extent and shape of the stability region give an idea of the step sizes that can be taken and the stability of the method. It is preferable for a method to have a larger stability region, particularly for stiff or highly oscillatory problems, because it means that larger step sizes can be taken.

The main requirement of method is that it is convergent, which means that the numerical solution converges to the exact solution as the step size approaches zero. Convergence is a crucial property for a numerical method. A fundamental result in numerical analysis is that, under certain conditions, consistency and stability lead to convergence. This connection underscores the need to establish both of these properties of a numerical method.

In light of these considerations, the development of effective and reliable numerical schemes requires a comprehensive framework that integrates error analysis, consistency, stability, and convergence. In this paper, we focus on the systematic investigation of these properties for a method designed to solve ODE's. The analysis contains the derivation of the LTE to establish the order of accuracy, verification of consistency, detailed stability analysis through the construction of the stability function, and characterization of the corresponding stability region. Finally, convergence is established by combining the results of consistency and stability.

2. MATERIALS AND METHODS

Many numerical methods have designed to attain the estimated consequences of IVP's for differential equation in nature ordinary. We can consider general function form:

$$\begin{cases} y'(x) = F(x_m, y_m), \quad x \in [a, b] \\ y(a) = \eta \end{cases} \tag{1}$$

Interval we have given as $[a, b]$ is parted into a m uniform parts, $x_k = a + kh, (k = 0,1,2,3, \dots m)$
, the step size is $h = x_{k+1} - x_k$. The to solve the function $y(x)$ in a discrete series equidistant node $x_{m-1} < x_m$ to attend approximate values $y_{m-1} < y_m$.
Generally, function of two space variable also familiar as a derivative of the dependent variable regarding to independent has been calculated through integrating (1)

$$y_{m+1} = y_m + \int_{x_0}^{x_0+h} F(x_m, y_m) dx$$

Many equations named differential don't reach to the solution such as particular and analytical. To deal with such difficulty new innovations take place in numerical analysis. Best advantage of numerical scheme is that these have better performance than analytical. Researchers day by day are trying to innovate many methods to obtain better consequences. But still huge number of work is needed to understand proper behaves of method. Researchers and scholars are out on the field of innovation to hone their skills. Many of them have also done a great job to construct and modified new methods. **Kandhro** [4] has developed accelerated method whose Iterative Integrator which is

$$s_1 = F(x_m, y_m)$$

$$s_2 = F\left(x_m + \tfrac{2}{3}h, y_m + \tfrac{2}{3}hs_1 + 2h^2 s_1 F_y\right)$$

$$s_3 = F\left(x_m + \frac{2}{3}h\,, y_m + \left\{\frac{1}{8}\left(\frac{1}{3}s_1 + 5s_2\right) - \frac{7}{4}hs_1F_y\right\}h\right) \quad (2)$$

$$y_{m+1} = y_m + \frac{h}{20}(5s_1 + 7s_2 + 8s_3)$$

This is newly developed an accelerated explicit scheme having three function evaluations per time step. Now error analysis.

3. ERROR ANALYSIS

It is main purpose to solve ordinary differential equation numerically to attain results which are as close as possible to the exact solution. Two sources of error which affect the accuracy of numerical method named as **round-off** (RO) **and truncation** (TR). RO round-off error takes the place when computers can only stock numbers with limited precision and **TR** arises because mathematical procedures are approximated (e.g., by deserting higher-order terms). An accuracy can totally rely on blunder what size of step size is taken. We consider Taylor's series for two variable up to the fourth power of step size h, we have

$$G(x_m, y_m) = y(x) + hF + \frac{h^2}{2}[F_x + F_yF] + \frac{h^3}{6}\begin{bmatrix} F_{xx} + 2\left(F_{xy} + \frac{1}{2}F_{yy}F\right)F \\ +(F_yF + F_x)\,F_y \end{bmatrix}$$

$$+\frac{h^4}{24}\begin{bmatrix} F_{xxx} + 3F(F_{xxy} + F_{xyy}F) + (5F_yF + 3F_x).F_{xy} \\ +F_{yyy}F^3 + 4F_yF_{yy}F^2 + 3F_xF_{yy}F + F_y^3F \\ +F_xF_y^2 + F_yF_{xx} \end{bmatrix} + O(h^5) \quad (3)$$

The local truncation error is an error which spawned in a unique step of the proposed improved scheme that is documented as L_{m+1} where

$$L_{m+1} = G(x+h) - y_{m+1} \quad (4)$$

Where $G(x+h)$ is the solution obtained by Taylors's Series and y_{m+1} is used as an approximate solution. Taylor series is utilized to expand these around x and similar terms collect in h. Now we expand proposed accelerated explicit Method present in eqn (2) upto h^4, we get

$$y_{m+1} = y(x) + hF + h^2\left[\frac{F_x}{2} + \frac{F_y}{2}F\right] + h^3\left[\frac{F_{xx}}{3!} + \frac{(2F_{xy} + F_{yy}F)F}{3!} + \frac{(F_yF + F_x)F_y}{3!}\right]$$
$$+h^4\left[\begin{array}{c}\frac{F_{xxx}}{27} + F\left(\frac{7}{135}F_{xyy} + \frac{1}{27}F_{yyy}F^2\right) + \frac{1}{9}F_xF_{xy} + \\ \frac{1}{18}F_yF_{xx} + F\left(\frac{1}{2}F_y^3 + \frac{8}{135}F_{xxy}\right) + \\ F^2\left(\frac{8}{135}F_{xyy} - \frac{3}{10}F_yF_{yy}\right) + \frac{1}{9}F_xF\,F_{yy} + \frac{2}{9}F_yFF_{xy}\end{array}\right] + O(h^5) \qquad (5)$$

Subtract (5) from (3). The proposed scheme has a local truncation error that is:

$$L_{m+1} = \frac{1}{216}\left[\begin{array}{c}F_{yyy}F^3 + \frac{(504F_yF_{yy} + 71F_{xyy})F^2}{5} + \\ \left(\frac{71}{5}F_{xxy} - 99F_y^3 + 3F_xF_{yy} - F_yF_{xy} - \frac{56}{5}F_{xyy}\right)F \\ +3\{(3F_y^2 + F_{xy})F_x - F_yF_{xx}\} + F_{xxx}\end{array}\right]h^4 + O(h^5) \qquad (6)$$

The local truncation error (LTE) often calls with name "discretization error per time step", which helps to evaluate the order of accuracy. From LTE it is observed, LTE has 4rth order then the proposed scheme in [4] has one less than the LTE in order. Therefore, the proposed scheme in [4] has third order of accuracy.

4. CONSISTENCY ANALYSIS

Definition 5.1 An increment function $\vartheta(x_m, y_m; h)$ of numerical method is entitled to be consistent, when

$$\lim_{h\to 0} \vartheta(x_m, y_m; h) = F(x_m, y_m)$$

Consistency of the numerical methods tells local truncation error (LTE) tends to zero as the step size decreases or $h \to 0$. Therefore, newly proposed scheme has increment function as

$$\vartheta(x_m, y_m; h) = \frac{1}{20}(5s_1 + 7s_2 + 8s_3)$$

By utilizing $\lim_{h\to 0}$ both sides

$$\lim_{h\to 0}\vartheta(x_m,y_m;h) = \lim_{h\to 0}\frac{1}{20}(5s_1+7s_2+8s_3)$$

$$= \lim_{h\to 0}\frac{1}{16}\left(5F(x_m,y_m)+7\left[F\left(x_m+\frac{2}{3}h\,,y_m+\frac{2}{3}hs_1+2h^2s_1F_y\right)\right]\right.$$
$$\left.+8\left[g\left(x_m+\frac{2h}{3}\,,y_m+\left\{\left(\frac{1}{24}s_1+\frac{5}{8}s_2\right)-\frac{7}{4}hs_1F_y\right\}h\right)\right]\right)$$

$$\lim_{h\to 0}\vartheta(x_m,y_m;h) = F(x_m,y_m)$$

Hence, the newly proposed scheme with at least **third order accuracy** is proven to be **consistent.**

5. LINEAR STABILITY ANALYSIS

Dahlquist's test problem is to be considered for verifying the stability of the Method in [4]

$\frac{dy}{dx} = q\,y(x)$; whereas $y(0) = \rho$

where q is a complex constant i.e, $q \in C$. After executing Method (2) on this test, we attain polynomial function acknowledged as stability function with linear region which displayed via unfilled area in fig I.

$s_1 = qy_m;$

$$s_2 = qy_m\left[1+\frac{2}{3}hq+2(hq)^2\right];$$

$$s_3 = qy_m\left[1+\frac{2}{3}hq-\frac{4}{3}(hq)^2+\frac{5}{4}(hq)^3\right];$$

Substitute all values in (5), polynomial form of stability function is derived as

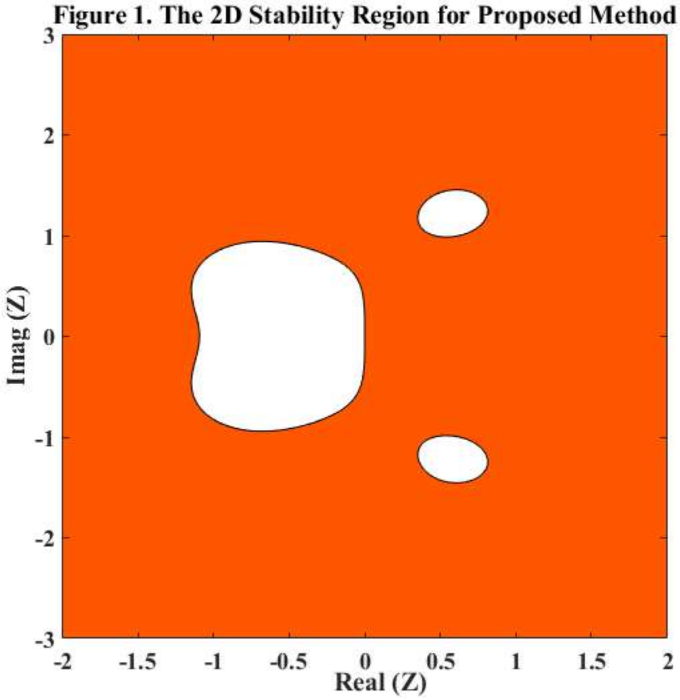
$$R(Z) = 1+z+\frac{z^2}{2!}\left(1+\frac{2}{3!}z+\frac{1}{2!}z^2\right) \text{ where } z = hq$$

This is Stability Function for accelerated proposed method in [4] which reflects the third-order accuracy of the method. When errors are introduced then linear stability is checked to know how numerical method behaves. In particular, it expresses that whether those errors **decay, remain bounded, or grow uncontrollably** during the computation. Stability controls error propagation.

6. REGION OF ABSOLUTE STABILITY

The region of the accelerated proposed method [4] is computed and visualized using MATLAB. A grid of complex values $z = x + iy$ is generated, and the stability function $R(z)$ is evaluated over this domain. The set of points satisfying $|R(z)| \leq 1$ is identified and plotted to illustrate the stability region in the complex plane.

Figure 1. The 2D Stability Region for Proposed Method



This stability **region** is a fundamental concept used to determine whether a numerical method produces **bounded solutions** when applied repeatedly. Figure I illustrates the two-dimensional region of absolute stability for the proposed numerical method in the complex plane. The real part $Re(z)$ is demonstrated on horizontal line of axis , while the imaginary part $Im(z)$ is demonstrated on vertical axis of line, where $z = hq$. The white (unshaded) regions indicate the set of values for which the stability condition $|R(z)| \leq 1$ is satisfied. These regions represent the domain when solution approximately remains stable and keep bounded. The orange (shaded) region corresponds to values of

z for which $|R(z)| \leq 1$ indicating instability, where numerical errors may grow exponentially.

7. CONVERGENCE OF THE METHOD

A method is to be numerically convergent when both consistency and stability justified. The necessary condition for convergence of a numerical method is **consistency**, which guarantees that the local discretization error vanishes as step-size very nearly approaches zero. However, convergence is achieved only when consistency is combined with stability. Therefore, above said words the accelerated proposed method is converges.

8. CONCLUSION

In this work, the construction and detailed mathematical analysis of a numerical method for solving initial value problems of ordinary differential equations. we theoretically examined well-organized third order numerical technique for IVP of ODE's including partial derivative which has enhanced its competency regarding truncation error. The main focus is not here to derive scheme but here is justify its theoretically behave which is analyzed by systematic investigation of error and convergence. The LTE has derived perfectly which endorsing accuracy by identify its order. Stability is investigated for knowing behavior of method. Stability region is envisioned to check numerical method possess a bounded solution when applied frequently and repeatedly. Convergence criteria theoretically discussed here by proving consistency and stability. Theoretically findings showing that the numerical method is stable, attains high accuracy and gives reliable performance. Therefore, method is efficient and applicable for extensive class of IVP arising in the area of ODE's.

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