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On Some Optimal Jarratt-Type Fourth Order Methods for Nonlinear Equations

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Abstract: *This research focuses on the problem of solving system of nonlinear equations by using numerical methods. We extend two-step Jarratt type iterative meth- ods with fourth order of convergence to solve system of nonlinear equations. The proposed methods does not require the evaluation of second or higher order Fréchet derivatives per iteration to proceed and reach fourth order of convergence. Convergence analysis of the new methods is also*

presented. Finally, numerical results illustrate the reliability and efficiency of the proposed methods.

Keywords *Nonlinear equations, Jacobian Matrix , Jarratt-type Methods, Efficiency Index*

Introduction

The development of iterative methods to solve nonlinear equations and system of nonlinear equations always remain important and challenging in the area of Numerical analysis. In the physical sciences, nonlinearity is every- where. Nonlinear equations have a number of applications in relativity, ecology, economics, gas dynamics, fluid mechanics, chemical reactions, transport theory, elasticity, biomechanics, and combustion and in many other phenomena. Therefore, modern mathematical research is extensively being devoted to the analysis of nonlinear phenomena and nonlinear systems.

The problem of finding the solution X∗ of a system nonlinear equations 5 (X∗) = 0 is a classical and important problem due to a large number of real world applications in all sciences and engineering disciplines. The function 5 : D c Rn → Rn is a sufficiently Fr´echet differentiable function in an open convex set D. In the last decade, several researchers have developed iterative schemes for solving system nonlinear equations to obtain the approximations of high accuracy; see, for example, [1, 2, 3, 5, 6, 7, 8, 9, 10, 11] and the references therein. Among all Newton's method is most famous and known method for finding the solution of the system 5 (X∗) = 0, for a given initial guess close enough to X∗[20], which is given as:

$$X^{(k+1)} = X^{(k)} — 5\,(X^{(k)})\,[5^{J}(X^{(k)})]^{—1}, h = 0, 1, 2, ..., (1)$$

Newton's method converges quadratically to X∗ ∈ D for a given initial guess close enough to the required root. In the expression (1), 5J(x(n)) is the Jacobian matrix of function 5 evaluated in the hth iterative step. Some high-order methods to find the solution of nonlinear system 5 (X) = 0 have been proposed in the literature based on Newton's scheme. The purpose of these iterative method was to increase the convergence speed and to boost the computational efficiency. The convergence order of any method can also be increased by the com- position technique. For example, the composition of two iterative methods with order r and s respectively provide us a method of order "rs". However, number of functional evaluation per iteration increase by using this technique. For the comparison of the different iterative methods, Ostrowski coined the concept of efficiency index of iterative methods, that is defined as for rth order method requiring ne functional evaluations per iteration then r ne is the efficiency index of that method [21]. Optimal Jarratt-type methods are the iterative methods that require one of evaluation of the function and two first-order derivative evaluations of the function to give order of convergence at least four. One should see [16, 12, 27], for the background of Jarratt-type methods.

In this paper, we propose two two-step iterative methods to find the ap- proximate solution of system of nonlinear equations, whose corresponding Jacobian matrix must be non-singular. The new schemes are an extension of existing two-step methods for single valued nonlinear equations. The significance of our presented schemes is that these schemes do not need the evaluation of the second, or higher order Fréchet derivatives per iterative step, and with the help of using only one functional evaluation and two evaluations of first-order Fréchet derivative per iterative step the fourth-order of convergence is obtained.

1.1 Basic Definitions And Concepts

Some basic definitions and concepts related to this are given in this section to enhance the readability.

Definition 1 AJgebraic PoJynomiaJ: An algebraic expression is an expression which consist of variable and constant are joined together by some algebraic operation such as addition (+), subtraction (—), multiplication (×), and division (÷). For example,

$$x^5 + 4x^4 + 2x^3 + 3x^2 + x + 3.$$

Definition 1·1 Non-Jinear Equation: An equation whose graph does nom not form an immediately line is referred to as non−linear equation. In non− linear equation the variables are either of the degree more than one or much less than one however never one. as an instance,

$$4x^2 + 2y — 1 = 0.$$

Definition 1.2 AJgebraic Equation: An equation in which the polynomial has a finite range of terms and equated to 0 is known as an algebraic equation.

$$s_n y^n + s_{n—1} y^{n—1} + s_{n—2} y^{n—2} + ... + s_1 y + s_0; s_n \neq 0$$

m here $,s_n, s_{n—1}, ...,$ are the actual coefficients of the polynomial.

Definition 1.3 Eero of an Equation: Zero is referred to as root of the equation f (y) = zero or the values for a variable y that fulfills the given expression.

Definition 1.4 MuJtipJe and SimpJe roots: A root m ith having a multiplicity m = 1, is known n as a simple root, i.e. f (x) = (x — 1) (x — 2) has the simple root at x0 = 1. A multiple root is that root having a multiplicity m ≥ 2 and also knomn as the multiple point or a repeated root. As in the example, the equation (x — 1)2 = 0,1 is as a multiple root.

Definition 1.5 Error : fhe difference betmeen a true value and an estimated or approximated value of the variable. Error= (frue value)−(approximated value). e = H — H∗

Definition 1.6 AbsoJute Error : Absolute error is defined as:

$$Eab = |H — H*|$$

While Eab is the absolute error & H∗ and H are approximated and true values respectively.

Definition 1.7 ÆeJatize Error: If H∗is approximated value of true value

H then the relative error is defined as:

$$H — H*$$

Where H /= 0

RE = H

Definition 1.8 System of NonJinear Equations (SN LE):fhe general form of a system of nonlinear equations is

$$f1(x1,x2, \qquad n) = 0$$

$$f2(x1,x2,......xn) = 0$$

$$.fn(x1,x2,......xn) = 0 \quad (1.2)$$

M here each function f k can be thought of as mapping a vector of the n−dimensional space Rn into the real line R. fhe system can alter− natively be represented by defining a functional 5,mapping Rn by:

$$X = (x1,x2, \qquad , xn)$$

$$5\ (x1, x2, ..., xn) = [f1(x1,x2,......xn), ..., fn(x1,x2, \quad xn)]T\ .$$

F the system (1.2) can also be m written as the form:

$$5\ (X) = 0,$$

M here, the functions $f1,f2,...fn$ are called the coordinate functions of 5 and X = (x1, x2, ..., xn)T .

Definition 1.9 Stopping Criterion: For the iterative technique computer successive approximation to the solution of a nonlinear trouble a prac− tical take a look at is manted to decide mhilst to stop the iteration. In ideal cases this test mould be measuring the space of ultimate nem release to the real ansmer homever this isn’t almays possible to do. Generally, the subsequent criterion is frequently used to stop the generation method:

$$X(n) — X(n—1) \ < \ x$$

$$X(n) < \ x;\ xn /= 0$$

$$5\ \ X(n) \ < \ x$$

Where x is the required tolerance value. Any of the above criterion can be used depending on the nature and behavior of the function.

Definition 1.10 Conzergence Order: Get a sequence X(n)}, $n \geq 0$, of real numbers generated by iterative methods converges to X∗. fhen, conver− gence is called of order r, $r \geq 1$, if there exists $M > 0$ and n0 such that

$$|X(n+1) — X* | \leq M |X(n) — X* |r, 6n \geq n0,$$

or

$$|e(n+1) | \leq M |e(n) |r, 6n \geq n0,$$

M here e(n) = X(n) — X∗. fhen me say that the sequence X(n+1) converges to X∗ mith the convergence order r and M is the asymptotic error constant.

Definition 1.11 Linear conzergence:For the sequence X(n)} of real num− bers mith X∗ as the limit, then the convergence order of the sequence is linear if there exists a constant $d < 1$ and an integer N in such a may that:

$$X(n) — X* \leq d |Xn — X*| , 6n \geq N$$

Definition 1.12 OptimaJ Order of Conzergence: A single step iterative method can have convergence order at most n requiring one functional evaluation and its n — 1 derivative evaluations. However, a multistep iterative method requiring n functional evaluations can achieve an optimal convergence order at most 2n—1.

Definition 1.13 ComputationaJ Conzergence Order: the formula to calculate computational order of convergence during the iteration process is
given as [21]:

$$m+1 \quad m (Em—Em—1)$$

$$sos = log, (Em—Em—1) (Em—1—Em—2)$$

Where, Xm, Xm—1, Xm—2 are the three consecutive iterations near to zero X∗.

Definition 1.14 Error Equation: If the error in ith iteration is e*k = z*k — α where z*k and α are the ith iterate and real root respectively. F then me can express an error equation as:

ek+1 = sek + O ek+1.

Definition 1.15 Efficiency Index: Efficiency index of an iterative scheme EI (IM) having the convergence order r, is described as follows:

EI(IM) = O ,

M where, is the number of evaluations required by an iterative method per iteration

2. Related work

Newton's iterative method is the most famous iterative method to find the approximate solution of the real or complex root β of a nonlinear equation *f* (x) = 0, which has the following iterative scheme [20]:

f (xk)

x= x —, h = 0, 1, 2, ... (2), k+1, k *f* r (x)

This method converges to quadratically in some neighborhood of the root β.

2.1 Jarratt-Type Methods for Solving Nonlinear Equa- tions

In this section, we study some Jarratt-type methods to find the approximate solution of nonlinear equations.

2.1.1 Jarratt-type Methods for Nonlinear Equations by Chun et al.

Changbum Chun in [14] developed an optimal family of fourth-order schemes from the Newton's iteration function as follows:

2*f* (xk) y = x —, h = 0, 1, 2, ..., (3), k 3*f* J(x)

f (xk) x= x — H (t (x)), (4)

where, k+1 k *f* J(x) k

f J(xk) — *f* J(yk), t(xk) = 2 *f* J(xk), (5))

If we take H(t) = 1 + 1(t) then (6) leads to an optimal fourth 1—t order Jarratt-type method [16] as follows,

f J(yk) — *f* J(xk) *f* (xk)

xk+1 = xk — [1 — JJ] J, (6)

f (yk) — *f* (xk) *f* (xk), where *f* (xk) y = x (7) 3*f* J(x)

If we take another H(t) = 1 + 9 (9) then (5) — (6) leads to another optimal fourth order Jarratt-type method [16]

x2 3ƒ (xk), k+1 = xk — w1(xk) — w2(xk) + ƒ J(x) + ƒ J(zk, (7))

where

w (x) = ƒ (xk), w

(x) = ƒ (xk) (8)

k ƒ J(x) k ƒ J(zk) , zk = xk — 3 w1(xk).

If we take another H(t) = 1 + t (t2) then (2.3)—(2.4) leads to another optimal 2 fourth order Jarratt-type method [16],

ƒ J(xk) — ƒ J(yk) ƒ J(xk) — ƒ J(yk)

ƒ (xk), xk+1 = xk + ([ƒ J(xk) ƒ J(xk))] ƒ J(x,) (9)

where

ƒ (xk) y = x, (10) k ƒ J (x)

2.1.2 Jarratt-type Methods for Nonlinear Equations by Khattri and Abbasbandy

Recently, another Jarratt-type method involving one evaluation of the func- tion and two evaluations of the first-order derivative has been developed by Khattri and Abbasbandy in [13], its iterative scheme is as follows.

ƒ (xk) y = x, k ƒ J(x) x= x ƒ (xk) [I + Σ α ƒ J(y)], (10)

where αk ∈ R e α1 = 21, α2 = 9, α3 = 15, α4 = 0, An example of the above Jarratt-type scheme is given as follows [13]

ƒ (xk) y = x, kƒ J(x)

xk+1 = xk — [1 + ƒ J(yk) 1 + ƒ J(xk) (ƒ J(yk)) +ƒ J(xk) ƒ J(yk) (ƒ J(xk))]. (11)

2.1.3 Jarratt-type Methods for Nonlinear Equations by Soleymani et al.

Soleymani et al. in [12] used weight function approach to build an optimal two-step class of fourth order Jarratt-type methods based on a simple change in Newton's scheme and weight functions at second step as follows:

ƒ (xk) y = x,   k ƒ J(x) ƒ (xk) x= x — [K(u) × G(v)],   (12)

k+1   k n ƒ J(x) + ƒ J(yk) r

where, u = ƒ(xn), v   = ƒ (yn) and K(u), G(v) are real-valued weight nƒ r(xn) nƒ r(xn)   n functions that should be chosen such that the fourth order of convergence is achieved. They proved the following theorem for the scheme (12):

Theorem 2.1 Get α be a simple root of ƒ:D ⊆ R → R in an open interval D. then the iterative family (2.13) has order of convergence at least four when

K(0) = 1, KJ(0) = KJJ(0) = 0, |K(3)(0)| ≤ +∞, G(1) = 1, GJ(1) = , GJJ) =, |G (1)| ≤ +∞,

and it satisfies the following error equation

e = (s+ s4, n+1 K (0) + 81 s2(297 + 32G (1))) en + O(en).

Following are some special cases of the scheme (14):

ƒ (xk) y = x,   ƒ J(x) ƒ (xk)

x = x [K × G], k+1 ƒ J(x) + ƒ J(yk)

where,

ƒ (xn) K = 1 +,ƒ J(xn)

ƒ J(yn) ƒ J(yn)

G1 = ƒ J(x) + 4   ƒ J(x) ,n

ƒ (xk) y = x, k ƒ J(x) ƒ (xk)   x= x — [K × G]   ,   (13)

k+1 = ƒ J(x) + ƒ J(yk)

where,

K1 = ,ƒ J(yn) ƒ J(yn)

G1=,ƒ J(xn)   ƒ J(x)

Another scheme presented by Soleymani et al. [12] is given as follows:

ƒ (xk)y= x,k   3 ƒ J(x)xk+1 = xk ƒ (xk)[K(n) × G(vn)][ƒ J(x) + ƒ J(y)], (14) where, u= ƒ(xn), v= ƒ (yn) and K(u), G(v) are real-valued weight ƒ r(xn) ƒ r(xn) functions that should be chosen such that the fourth order of convergence is achieved. They proved the following theorem for the scheme (14):

Theorem 2.2 Get f:D ⊆ R → R has a simple root α in an open interval D. fhen the class of iterative methods (2.16) is of fourth order convergent mhen

K(0) = 1, KJ(0) = KJJ(0) = 0, |K(3)(0)| ≤ +∞, G(1) = 1, GJ(1) = —1 , GJJ(1) = 1, |G(3)(1)| ≤ +∞

and it satisfies the following error equation

e = (s+ s4 n+1(0) + 81 s2(en + O(en).

Following are some special cases of the scheme (16):

f (xk) y = x, f J(x)

f (xk) x = x — [K × G], (17)

k+1f J(xk) f J(yk) where, K1 = f J(yn) f J(yn)

G1 = +f J(xn), f J(xn)

2.1.4 Jarratt-type Methods for Nonlinear Equations by Junjua et al.

Recently, Junjua et al. in [17] developed a family of Jarratt-type methods for solving nonlinear equations as follows:

f (xk) y = x — α f J(x) f (xk) x = x — [S(u) × T (v)], (19)

f J (y) where uk f (xh) f r(xk) = fr(yh) , and S (u h) and T (vk) are the real valued weight functions that should be selected in such a way that the scheme (18) possess the order of convergence at least four. The following theorem show the analysis of convergence of the above scheme.

Theorem 2.3 Get x∗ ∈ D be a simple root of the sufficiently differentiable function f: I ⊆ R → R in an open interval D containing x∗. then for a = 2, the scheme (2.18) has optimal convergence order four under the following conditions on weight functions:

S (1) = SJ(1) =, SJJ 4 (1) =, |S(1)| ≤ ∞, T (0) = 1, T J(0) = T JJ(0) = 0, |T JJJ(0)| ≤ ∞ ,(20)

and it satisfies the error equation given by:

e = (k+1(1))s2 — s2s3 + s4 T (0))ek + O(ek), where sk f (h)(x) k!f r(x)

Proof. Let the error at h tk iterative step is ek = xk — x∗. The Taylor's expansion of ƒ (xk) about the root x∗is given as follows:

$$ƒ (xk) = ƒ J(x∗)(ek + s2e2 + s3e3 + s4e4) + O(e5), \quad (21)$$

where sk = ƒ (x) , h ≥ 2. The first derivative in the first step of our scheme can be calculated as:

$$ƒ J(xk) = ƒ J(x∗) (1 + 2s2ek + 3s3e2 + 4s4e3) + O(e4). \quad (22)$$

Using (21) and (22) in the first step of (18), the Taylor's expansions for yk and ƒ J(yk) are given by:

$$yk= r + (1 — α)ek + αs2e2 α(—2s3 + 2s2)e3 — α(s2 —3s3 + 4s2 — 3s4 +4s2s3 + 2(3s3 — 4s2)s2)e4 + O(e5), \quad (23)$$

$$ƒ J(yk) = ƒ J(x∗)(1 + 2s2(1 — α)ek + (2αs2 + 3s3(1 — α)2)e2 + (6s3(1 — α)αs2 —2s2α(—2s3 + 2s2) + 4s4(1 — α)3e3)) + O(e4). \quad (22)$$

Similarly, for computation of the second step of the scheme (18), we can obtain:

$$ƒ (xk) = e + (—2s (1 — α) + s) e2 + (—2s2(1 — α) — 2αs2 — 3s (1 — α)2 ƒ J(yk) 4s2(—1 + α)(1 — α) + s3)e3 + ... + O(e5). \quad (23)$$

By using (2.20), (2.21) and (2.18), we get

$$Uk = ek — s2e2 + (—2s3 + 2s2)e3 + (s2(—3s3 + 4s2) — 3s4 + 4s2s3 + 2(3s34s2)s2)e4 + O(e5), \quad (24)$$

$$vk = 1 + (—2s2 + 2s2(1 — α))ek + (—3s3 + 4s2 + 2αs2 + 3s3(1 — α)24s2(1 — α))e2 + \cdot\cdot\cdot + O(e5). \quad (25)$$

Again using Taylor's expansion, we have:

$$S(uk) = S(0) + S (0)ek + (—S (0)s2 + 2 S (0))ek + \cdot\cdot\cdot + O(ek) \quad (26)$$

and

$$T (vk) = T (1) — 2T J(1)αs2ek + (6T J(1)αs2 — 6T J(1)αs3 + 3T J2s3 +2T JJ2s2)e2 + \cdot\cdot\cdot + O(e5). \quad (27)$$

Substituting α = (19), (26) and (25) in (2.18) and using (19) we have: x = x∗ + (k+1 (1)) s2 s2s3 + 9 s4 S(0))ek + O(ek)

which gives the following error equation:

e = (k+1(1))s2 — s2s3 + 9 s4 S (0))ek + O(ek). (28)

Hence, it can be seen that the new scheme has optimal fourth order convergence. A special case of weight functions for the family (29) is given as follows:

H(u) = ƒ r(yh)ƒ r(yh) G(vk) = 1 (29)

Thus, another Jarratt type fourth order method is given as follows:

Y = xƒ (xk) ,k ƒ J(x) x= x ƒ r(yh)

ƒ r(yh) ƒ (xk) . (2.30)

2.2 Jarratt-Type Methods for Solving System of Non- linear Equations

A brief review of some iterative methods to solve system of nonlinear equations is presented in the section. These methods are the extensions and generalizations of the schemes discussed in the Section 2.1 for single valued nonlinear equations.

The extension of the schemes (5)-(6) proposed by Khairallah and Hafiz [27] to solve the system of nonlinear equations for a given initial approximation X0 is given as:

Z(k) = X(k) (X (k)) 1(X (k)), X(k+1) = X(k) [I(Y (k)))J(X (k)))1 (k) J(X (k)r)))](Z (k)) (X (k)), (31)

The extension of the scheme (7)-(8) proposed by Khairallah and Hafiz [27] for the solution of system of nonlinear equations for a given initial ap- proximation X0 is given as:

Z(k) = X(k) (X(k))(X(k)),X(k+1) = X(k)r(k) (X)+ (Z 2)] (xk)+ [((X) (Z)] (xk), (32) The extension of the scheme (2.12) proposed by Khairallah and Hafiz [27] to solve the system of nonlinear equations for a given initial approximation X^0 is given as: The following theorem was proved in [27].

Theorem 2.4 Get X∗ be a root of 5 :⊆ Rn → Rn in an open interval D, 5 is sufficiently Frechet differentiable function. If the initial guess X(0) is close enough to the root X∗, then the tmo−step iterative scheme given by (2.34) has convergence order at least 4.

Proof. Consider to

$$y_k = x_k - \frac{2f(x_k)}{3f'(x_k)}, k = 0, 1, 2, ..., \tag{2.2}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} H(t(x_k)), \tag{2.3}$$

where,

$$t(x_k) = \frac{3}{2} \frac{f'(x_k) - f'(y_k)}{f'(x_k)}, \tag{2.4}$$

If we take $H(t) = 1 + \frac{1}{2}(\frac{t}{1-t})$ then (2.3) − (2.4) leads to an optimal fourth order Jarratt-type method [16] as follows,

$$x_{k+1} = x_k - [1 - \frac{3}{2} \frac{f'(y_k) - f'(x_k)}{3f'(y_k) - f'(x_k)}] \frac{f(x_k)}{f'(x_k)}, \tag{2.5}$$

where

$2f(x_k)$

where

$$\omega_1(x_k) = \frac{f(x_k)}{f'(x_k)}, \omega_2(x_k) = \frac{f(x_k)}{f'(z_k)}, z_k = x_k - \frac{2}{3}\omega_1(x_k). \tag{2.8}$$

If we take another $H(t) = 1 + \frac{t}{2}(\frac{t^2}{2})$ then (2.3) − (2.4) leads to another optimal fourth order Jarratt-type method [16],

$$x_{k+1} = x_k - [1 + \frac{3}{4} \frac{f'(x_k) - f'(y_k)}{f'(x_k)} + \frac{9}{8}(\frac{f'(x_k) - f'(y_k)}{f'(x_k)})^2] \frac{f(x_k)}{f'(x_k)}, \tag{2.9}$$

where

$$y_k = x_k - \frac{2f(x_k)}{3f'(x_k)}, \tag{2.10}$$

New Jarratt-Type Methods for Solving System of Nonlinear Equations

we extend the Jarratt-type methods developed by Soleymani et al. [12] for the solution of system of nonlinear equations. Convergence analysis and numerical results are presented for the comparison of new and existing methods. We extend of the scheme (2.17) to solve system of nonlinear equations for a given initial approximation X0 as follows:

$$y_k = x_k - \frac{2f(x_k)}{3f'(x_k)},$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}[I + \sum_{j=1}^{4} \alpha_k (\frac{f'(y_k)}{f'(x_k)})^j], \qquad (2.11)$$

where $\alpha_k \in \mathbb{R} \ni \alpha_1 = \frac{21}{8}, \alpha_2 = \frac{-9}{2}, \alpha_3 = \frac{15}{8}, \alpha_4 = 0,$

An example of the above Jarratt-type scheme is given as follows [13]

Results & Remarks

$$y_k = x_k - \frac{2f(x_k)}{3f'(x_k)},$$

3.1 Numerical Results

$$x_{k+1} = x_k - [I + \frac{21}{8}(\frac{f'(y_k)}{f'(x_k)})^1 + \frac{-9}{2}(\frac{f'(y_k)}{f'(x_k)})^2 + \frac{15}{8}(\frac{f'(y_k)}{f'(x_k)})^3]. \qquad (2.12)$$

We now, study the numerical behavior of the proposed methods (3.1) (NM1) and (3.9) (NM2) for solving system of non-linear equations with the methods presented by Khairallah and Hafiz (2.32)-(2.34) denoted by KH1,KH2 and Junjua's methods (2.44) (JM). We have used a precision of 500 decimal digits for solving system of nonlinear equations on the programing package Maple16. We have used the following stopping criterion to obtain the results.

$$\left\|\mathbf{F}(\mathbf{X}^{(k-1)})\right\|_\infty + \left\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\right\|_\infty < 10^{-100}$$

Tables 1-3 show absolute values of the difference between two consecutive approximations of the root and absolute functional values at the approximate

Example 1 *We consider the following system*

$$\begin{aligned} x_1^2 - x_2 - 19 &= 0, \\ \frac{x_2^3}{6} - x_1 + x_2 - 17 &= 0, \end{aligned}$$

with the exact solution $X^* = (5,6)^T$. *We used the initial approximation* $X^0 = (5.1, 6.1)$ *to start the iterative process.*

Example 2 *We take another system of nonlinear equations as:*

$$\begin{aligned} \cos x_2 - \sin x_1 &= 0, \\ x_3^{x_1} - \frac{1}{x_2} &= -0, \\ e^{x_1} - x_3^2 &= 0. \end{aligned}$$

The solution of above system is $X^* \approx (0.909569, 0.661227, 1.575834)^T$. *We choose the initial vector* $X^0 = (1, 0.5, 1.5)^T$ *to start the iterative process.*

Example 3

$$\begin{aligned} x_1x_3 + x_4(x_1 + x_3) &= 0, \\ x_1x_2 + x_4(x_1 + x_2) &= 0, \\ x_2x_3 + x_4(x_2 + x_3) &= 0, \\ x_1x_2 + x_1x_3 + x_2x_3 &= 1. \end{aligned}$$

We find the approximate solution of above system using the starting vector $X^0 = (0.5, 0.5, 0.5, -0.2)^T$. *The exact solution of above system is*

Nu ... 2 perform

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Table 1: Comparison of Various Iterative Methods for Example 1

Table 1: Comparison of Various Iterative Methods for Example 1

Methods	Iterations	$\left\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\right\|_\infty$	$\|F(X^{(k-1)})\|_\infty$
KH1	5	$2.59\ 10^{-103}$	$4.64\ 10^{-102}$
KH2	5	$1.18\ 10^{-376}$	$2.13\ 10^{-375}$
KH3	4	$2.59\ 10^{-103}$	$4.64\ 10^{-102}$
JM	4	$7.10\ 10^{-108}$	$1.2\ 10^{-106}$
NM1	4	$3.10\ 10^{-109}$	$3.2\ 10^{-107}$
NM2	4	$5.10\ 10^{-110}$	$5.2\ 10^{-109}$

Table 2: Comparison of Various Iterative Methods for Example 2

Methods	Iterations	$\left\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\right\|_\infty$	$\|F(X^{(k-1)})\|_\infty$
KH1	6	$1.00\ 10^{-299}$	$1.02\ 10^{-299}$
KH2	6	$2.76\ 10^{-210}$	$2.79\ 10^{-210}$
KH3	6	$1.00\ 10^{-299}$	$1.02\ 10^{-299}$
JM	5	$1.18\ 10^{-356}$	$1.23\ 10^{-356}$
NM1	5	$3.18\ 10^{-360}$	$3.23\ 10^{-360}$
NM2	5	$2.18\ 10^{-361}$	$2.23\ 10^{-361}$

Table 3: Comparison of Various Iterative Methods for Example 3

Methods	Iterations	$\left\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\right\|_\infty$	$\|F(X^{(k-1)})\|_\infty$
KH1	5	$3.49\ 10^{-238}$	$4.02\ 10^{-238}$
KH2	5	$1.30\ 10^{-198}$	$1.30\ 10^{-198}$
KH3	5	$3.49\ 10^{-238}$	$4.02\ 10^{-238}$
JM	4	$1.02\ 10^{-256}$	$1.02\ 10^{-256}$
NM1	4	$3.02\ 10^{-258}$	$3.02\ 10^{-258}$
NM2	4	$2.02\ 10^{-257}$	$2.02\ 10^{-257}$

3.2 Concluding Remarks

In this research we have studied the problem of solving system of nonlinear equations numerically. We extend existing two-step Jarratt type iterative methods with fourth order of convergence to solve system of nonlinear equations. The proposed methods does not require the evaluation of second or higher order Fréchet derivatives per iteration to proceed and reach fourth order of convergence. Convergence analysis of the new methods is also presented. Finally, numerical results illustrate the robustness and efficiency of the proposed methods

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