



A Modified SEIR Model for Poverty Dynamics: Analysis and Stability

Zohaib Ali

zohaib.ali@usindh.edu.pk

Institute of Mathematics & Computer Science, University of Sindh, Jamshoro

Hamadullah Bhutto

Hamad.bhutto@usindh.edu.pk

Institute of Mathematics & Computer Science, University of Sindh, Jamshoro

Irfan Ahmed Jatoi

irfan.jatoi@usindh.edu.pk

Department of Basic Sciences & Related Studies, Mehran University of Engineering & Technology, Jamshoro

Abdul Rafiu Alias Furkan

fmemon022@gmail.com

Subject Specialist GBHSS Mithani, Naushahro Feroze, Sindh Education and Literacy Department, Government of Sindh

Corresponding Author: zohaib.ali@usindh.edu.pk

Abstract: This study presents a modified SEIR model to analyze poverty dynamics in a population. The total population is divided into four compartments: financially stable, economically vulnerable, poor, and recovered individuals. Poverty transmission occurs through interaction between stable and poor populations, while relapse of recovered individuals is incorporated via economic shocks. The model includes demographic recruitment and natural exit rates, making it realistic for long-term population studies. Analytical techniques such as positivity of solutions, invariant region, existence and uniqueness, disease-free equilibrium, and stability analysis are applied. The basic reproduction number \mathcal{R}_0 is derived to measure the potential for poverty persistence. Numerical simulations illustrate the impact of varying transmission, progression, and recovery rates on poverty prevalence. Local stability of the disease-free equilibrium is established via eigenvalue analysis and upper-triangular reduction. Global stability is proved using a Lyapunov function. The findings provide insight into policy interventions to reduce economic vulnerability.

Keywords: Poverty, SEIR Model, Disease Free Equilibrium, Basic Reproduction Number, Eigenvalues, Lyapunov Function

1 Introduction

Poverty has been among the most burning social issues in the world, where millions of people and families have been affected, and their access to basic services like food, health services and



education has been greatly restricted [1,2]. Poverty dynamics in developing countries are complex interactions between economic, social and demographic factors. Conventional statistical approaches offer scanty information on how poverty changes over time and how it is fed back in a population. Mathematic modeling provides a strong framework of this understanding the spread and persistence of poverty through nonlinear interactions, transitions between various socioeconomic states. SEIR has been extensively applied to socioeconomic research as well as in epidemiology, and other deterministic compartmental models have been utilized widely in both fields [3,4]. The populations may be represented as separate compartments by these models and the researchers can quantify the flow in and out of the stable, vulnerable, poor, and recovered people with time. The model achieves a natural long-term behavior of poverty by adding demographic recruitment, natural exit and relapse dynamics. In this paper, a modified SEIR framework of poverty has been developed to offer policy recommendations and analytical findings. The model is analytical and quantitative in its assessment of the circumstances in which poverty will remain or fade in a population. The outcomes of the findings are expected to be used to justify specific interventions and sustainable development plans.

Poverty mathematical models provide the possibility to examine both direct and indirect impacts of socioeconomic intervention. Past researches have studied poverty in terms of agent-based simulation, stochastic analysis and deterministic models. Although agent-based models can be used to study individual heterogeneity, deterministic models are analytic, permit stability and threshold studies [5,6]. Through the modification of the SEIR model, which has been traditionally applied to the modeling of infectious diseases, it will be possible to model the transmission of poverty and to describe it as a similar process of transmission of disease. This is because financially stable people may be weakened as they interact with the poor populations and also because vulnerable people may degenerate into poor people just as epidemiology shows members of the exposed population transforming into members of the infectious category. Even recovered people may relapse as a result of economic shocks or policies failure, which is why it is important to take feedback loops into the system. This methodology allows determining such central measures as the basic reproduction number, which is an indicator of the poverty persistence threshold, as the basic reproduction number, denoted as \mathcal{R}_0 . The model takes into consideration both the relapse mechanisms and natural population dynamics hence presents a rich perspective of the socioeconomic system. It is also suitable to use in sensitivity analysis where variables can be determined to have the strongest effects in poverty reduction. The framework is thus analytically sound, as well as practically useful in policy planning.

When developing the modified poverty SEIR model, the population is stratified into four compartments [7-9], which include those that are financially stable (S), economically vulnerable (E),

poor (I), and those that are recovered (R). People join the system by being recruited into the stable class and this is a measure of new births or entries into the economy. The transmission of poverty is realized when stable and poor people are in contact with each other and at a sharing ratio between these populations, which is a social and economic impact. At a predetermined pace, vulnerable people graduate to the poor category keeping the risk of loss of financial security. Interventions, employment, or social support may help poor individuals to recover at a recovery rate and move to the recovered class. Recovered population is however vulnerable to relapse because of economic shocks or bad policies which is shown by a shift of R to E . Each of the compartments is subject to natural exit, which is a form of mortality or migration, and keeps the overall population fixed. This structure enables the model to reflect the two aspects of society; forward movement and relapse of the dynamics between poverty and development. The model is therefore a realistic and flexible construct to consider the long-term socioeconomic trends. It also allows policy makers to model the impacts of the intervention strategies (in the long term).

The model is analytically studied by deriving equilibrium points and analysing stability and computing the basic reproduction number, \mathcal{R}_0 . The disease-free equilibrium is a situation in which there is no poverty, or poverty is minimal whereas the endemic equilibrium is a situation of chronic poverty. The analysis of local stability is performed by calculating the Jacobian at the equilibrium points and the analysis of eigenvalues, whereas the global stability can be determined by Lyapunov functions. The positivity of solutions and the fact that there is a region of feasibility is also proved which ensures that the model is both biologically and socially interesting. Basic reproduction number \mathcal{R}_0 measures the ability of the poverty to continue or increase among the population. When the value of \mathcal{R}_0 is below 1, then interventions are enough to avoid the process of spreading poverty and then the system will automatically reach an equilibrium where there is no poverty. Where the \mathcal{R}_0 exceeds one, poverty will stay and even turn into a permanent condition, which means that specific steps are to be taken. Sensitivity analysis is used to determine which parameters have the greatest impact on the value of \mathcal{R}_0 , which is then used to optimally allocate resources. The analytical results are supplemented by the numerical simulations that are used to demonstrate the dynamic interactions between compartments.

This is not merely a theoretical value of this modeling approach, which is why it is applicable in policy and social planning. The different intervention situations can be modeled by the policymakers to test the efficiency of a social program, employment plans, or economic support programs. The model allows one to determine the effect of variations in the transmission, progression, recovery and relapse rates on poverty rates at any given time. Also, the framework can be adapted to the specific regional/national environments, which is premised on the local demographics, economy, and local

social policies. This is because such models are best suited in a case like Pakistan where the poverty and economic inequality remains high. They provide quantitative guidelines that can be used to work on strategies that will reduce vulnerability and enhance the goals of sustainable development. Analytical and numerical knowledge will also be combined, which will result in the evidence-based nature of the interventions and the maximum impact. The model is hence an intermediate between the policy in theory and the policy in action.

Finally, the research paper contributes to accumulating knowledge of socioeconomic modelling by applying epidemiological theories to the relationships between poverty and relationships. The adapted SEIR model is strict, natural, and adaptable to allow the development, as well as the relapsing prospects of the phenomenon of poverty in a population. The model determines the important thresholds, stability conditions, and determining parameters with the assistance of analytical derivations and numerical simulations. The results emphasize the use of early intervention, recovery programs and social protection policies. The model assists in informing the resource allocation and long-term planning through highlighting the parameters with the greatest influence on the dynamics of poverty. Ultimately, it can provide a systematic process of research and reduction of poverty in a quantitative systematic manner. The paper demonstrates that mathematical modeling could be applied as a tool which would assist to make economic and social choices on developing countries.

2 Methodology

The paper constructs a deterministic compartmental model to examine the poverty dynamics with a long run SEIR model. The overall population is further broken down into four mutually exclusive strata including financially stable: $S(t)$, economically vulnerable: $E(t)$, poor: $I(t)$, and recovered: $R(t)$. To get into the system, people are recruited into the stable class. Socioeconomic interaction between the stable and poor people leads to poverty transmission. A relapse mechanism is included to capture economic shocks that can lead the recovered individuals into the state of vulnerability. Natural exit has an equal impact on each and every compartment. The model is presented in the form of nonlinear equations of ordinary differential equations. Equilibrium analysis, positivity of solutions and stability theory are some of the analytical methods that can be used to analyze the long-term behavior of the system. Numerical simulations can also be used to demonstrate the impact of the variation of parameter on poverty prevalence.

2.1 Model Formulation

The total population at time t is given by

$$N(t) = S(t) + E(t) + I(t) + R(t). \quad (1)$$

Individuals enter the stable class at rate Λ . Poverty spreads when stable individuals interact with poor individuals at transmission rate β . Vulnerable individuals progress to poverty at rate σ , while poor individuals recover at rate γ . Natural removal occurs from all classes at rate μ . Recovered individuals may relapse into vulnerability due to economic instability at rate θ . Under these assumptions, the governing system is

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta \frac{SI}{N} - \mu S, \\ \frac{dE}{dt} &= \beta \frac{SI}{N} + \theta R - (\sigma + \mu)E, \\ \frac{dI}{dt} &= \sigma E - (\gamma + \mu)I, \\ \frac{dR}{dt} &= \gamma I - (\mu + \theta)R. \end{aligned} \quad (2)$$

Substituting $b_0 = \sigma + \mu$, $b_1 = \gamma + \mu$ and $b_2 = \mu + \theta$ in the model (2), the model will be obtained as:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta \frac{SI}{N} - \mu S, \\ \frac{dE}{dt} &= \beta \frac{SI}{N} + \theta R - b_0 E, \\ \frac{dI}{dt} &= \sigma E - b_1 I, \\ \frac{dR}{dt} &= \gamma I - b_2 R. \end{aligned} \quad (3)$$

2.2 Flow Diagram

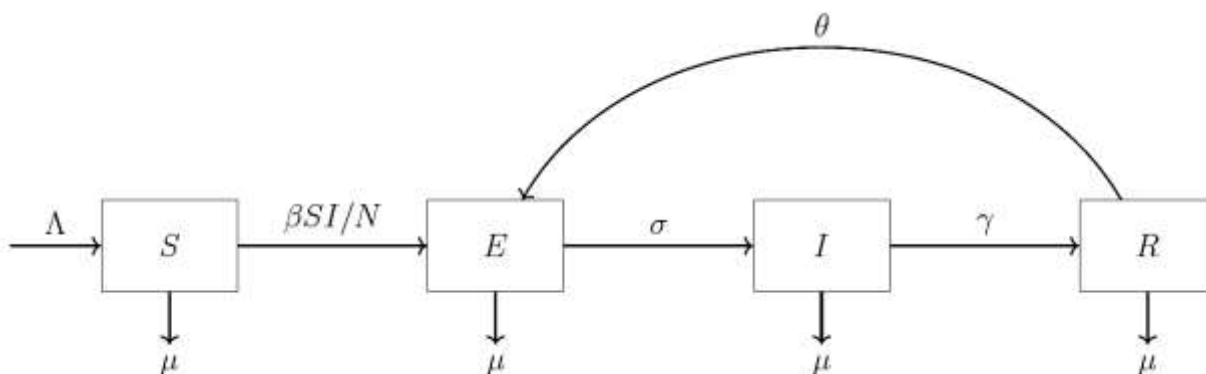


Figure 1: Flow diagram of the modified poverty SEIR model

2.3 Parameter Description

Table 1: Description of Parameters Used in the Model (2)

Parameter	Description
Λ	Recruitment rate into stable population
β	Poverty transmission rate
σ	Vulnerable to poor progression rate
γ	Recovery from poverty rate
μ	Natural exit rate
θ	Relapse rate from recovered to vulnerable

2.4 State Variables

Table 2: Description of State variables of the model (2)

Variable	Description
$S(t)$	Financially stable population
$E(t)$	Economically vulnerable population
$I(t)$	Poor population
$R(t)$	Recovered population

3 Mathematical Analysis of the Model

3.1 Positivity of Solutions

Theorem (Positivity). Let the initial conditions satisfy

$$S(0) > 0, \quad E(0) > 0, \quad I(0) > 0, \quad R(0) > 0$$

Then the solutions $S(t), E(t), I(t), R(t)$ of the system remain non-negative for all $t > 0$ [10].

Proof. Consider the first equation

$$\frac{dS}{dt} = \Lambda - \beta \frac{SI}{N} - \mu S. \tag{4}$$

Since all parameters and state variables are nonnegative,

$$\frac{dS}{dt} \geq -(\beta I/N + \mu)S. \tag{5}$$

This implies

$$S(t) \geq S(0)e^{-(\beta I/N + \mu)t} \geq 0. \tag{6}$$

Thus $S(t)$ remains positive. Similar arguments applied to $E(t), I(t), R(t)$ show that all solutions remain nonnegative for all $t > 0$. Hence the system is positive.

3.2 Invariant Region

Theorem (Invariant Region). The feasible region

$$\Omega = \left\{ (S, E, I, R) \in \mathbb{R}_+^4 : N(t) \leq \frac{\Lambda}{\mu} \right\}. \tag{7}$$

is positively invariant for the system [11].

Proof. Adding all equations gives

$$\frac{dN}{dt} = \Lambda - \mu N. \quad (8)$$

This linear differential equation has solution

$$N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}). \quad (9)$$

Taking limit as $t \rightarrow \infty$ yields

$$N(t) \leq \frac{\Lambda}{\mu}. \quad (10)$$

Hence any solution starting in Ω remains in Ω for all $t > 0$. Therefore Ω is positively invariant.

3.3 Existence and Uniqueness via Lipschitz Condition

Theorem. Let the initial conditions

$$(S(0), E(0), I(0), R(0)) \in \mathbb{R}_+^4$$

be given. Then the poverty SEIR model admits a unique solution for all $t > 0$.

Proof. Define the vector function

$$X = (S, E, I, R)^T \quad \text{and} \quad \frac{dX}{dt} = F(X),$$

where $F(X)$ represents the right-hand side of the system. Each component of $F(X)$ consists of polynomial and rational expressions in the variables S, E, I, R . Since $N = S + E + I + R > 0$, these

functions are continuously differentiable in any biologically feasible region. Therefore the partial derivatives of F are continuous and bounded in every closed bounded subset of \mathbb{R}_+^4 . Hence F satisfies the Lipschitz condition

$$\|F(X_1) - F(X_2)\| \leq L \|X_1 - X_2\|. \quad (11)$$

for some constant $L > 0$. By the Picard–Lindelöf theorem, a unique local solution exists. Because the invariant region ensures boundedness of solutions, this local solution extends to all $t > 0$. Thus the system possesses a unique global solution.

3.4 Disease-Free Equilibrium and Basic Reproduction Number

Disease-Free Equilibrium (DFE).

The disease-free equilibrium is obtained by setting

$$E = 0, \quad I = 0, \quad R = 0$$

and solving the system at steady state.

From

$$\frac{dS}{dt} = \Lambda - \mu S = 0. \quad (12)$$

we obtain

$$S^* = \frac{\Lambda}{\mu}. \quad (13)$$

Hence the disease-free equilibrium is

$$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right). \quad (14)$$

Basic Reproduction Number R_0 .

The basic reproduction number is computed using the next-generation matrix method [12-14]. Consider the infected subsystem consisting of compartments E and I .

New infection terms:

$$F = \begin{pmatrix} \beta \frac{SI}{N} \\ 0 \end{pmatrix}. \tag{15}$$

Transition terms:

$$V = \begin{pmatrix} b_0 E - \theta R \\ b_1 I - \sigma E \end{pmatrix}. \tag{16}$$

At the disease-free equilibrium E_0 , we substitute

$$S = \frac{\Lambda}{\mu}, \quad E = I = R = 0.$$

The Jacobian matrices of \mathcal{F} and \mathcal{V} evaluated at E_0 are

$$F = \begin{pmatrix} 0 & \beta \frac{\Lambda}{\mu N} \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} b_0 & 0 \\ -\sigma & b_1 \end{pmatrix}.$$

The basic reproduction number is the spectral radius of

$$FV^{-1}. \tag{17}$$

After computation, we obtain

$$R_0 = \frac{\beta\sigma}{b_0 b_1}. \tag{18}$$

Interpretation.

- If $R_0 < 1$, poverty will eventually disappear.
- If $R_0 > 1$, poverty persists and may become endemic.

3.5 Local Stability of Disease-Free Equilibrium

Theorem (Local Stability of DFE). The disease-free equilibrium

$$E_0 = (S^*, E^*, I^*, R^*) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right). \tag{19}$$

is locally asymptotically stable if $R_0 < 1$.

Proof.

The Jacobian matrix of the system (2) at DFE is obtained as:

$$J(E_0) = \begin{pmatrix} -\mu & 0 & -\beta & 0 \\ 0 & -b_0 & \beta & \theta \\ 0 & \sigma & -b_1 & 0 \\ 0 & 0 & \gamma & -b_2 \end{pmatrix}. \tag{20}$$

First eigenvalue from above matrix is:

$$\lambda_1 = -\mu < 0. \tag{21}$$

is an eigenvalue.

After removing the S row and column corresponding to λ_1 , the remaining 3×3 matrix is:

$$J(E_0) = \begin{pmatrix} -b_0 & \beta & \theta \\ \sigma & -b_1 & 0 \\ 0 & \gamma & -b_2 \end{pmatrix}. \tag{22}$$

The above matrix is reducing into upper triangular matrix as:

$$J(E_0) = \begin{pmatrix} -b_0 & \beta & \theta \\ 0 & -b_1 + \frac{\sigma\beta}{b_0} & \frac{\sigma\theta}{b_0} \\ 0 & 0 & -b_2 - \frac{\gamma\sigma\theta}{b_0[-b_1 + \frac{\sigma\beta}{b_0}]} \end{pmatrix}. \tag{23}$$

Since the above matrix is upper triangular matrix and its eigenvalues are the diagonal elements which are obtained as:

$$\begin{aligned}\lambda_2 &= -b_0, \\ \lambda_3 &= -b_1 + \frac{\sigma\beta}{b_0}, \\ \lambda_4 &= -b_2 - \frac{\gamma\sigma\theta}{b_0\left(-b_1 + \frac{\sigma\beta}{b_0}\right)},\end{aligned}\tag{24}$$

Since λ_2 is negative and λ_3 and λ_4 are also negative after using the value of \mathcal{R}_0 and simplification yields:

$$\begin{aligned}\lambda_3 &= b_1(\mathcal{R}_0 - 1), \\ \lambda_4 &= -\left(b_2 + \frac{\gamma\theta\mathcal{R}_0}{\beta(\mathcal{R}_0 - 1)}\right),\end{aligned}\tag{25}$$

All eigenvalues have negative real parts which proves the local stability of DFE [15-18].

3.6 Global Stability of Disease-Free Equilibrium

Theorem (Global Stability). If $R_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable in the feasible region Ω .

Proof. Consider the Lyapunov function

$$V(E, I) = \sigma E + b_0 I.\tag{26}$$

Clearly $V > 0$ for $E, I > 0$ and $V = 0$ only at the DFE.

Differentiating along solutions,

$$\dot{V} = \sigma \dot{E} + b_0 \dot{I}.\tag{27}$$

Substituting \dot{E} and \dot{I} values from model equations,

$$\dot{V} = \sigma \left(\beta \frac{SI}{N} + \theta R - b_0 E \right) + b_0 (\sigma E - b_1 I). \quad (28)$$

After simplification and eliminating a nonnegative term, the equation is reduced as:

$$\dot{V} \leq \sigma \beta \frac{SI}{N} - b_0 b_1 I. \quad (29)$$

Since at DEF $S = N$ and after simplification, above equation is obtained as:

$$\dot{V} \leq b_0 b_1 (\mathcal{R}_0 - 1) I. \quad (30)$$

If $\mathcal{R}_0 < 1$, then $\dot{V} \leq 0$ with equality only when $I = 0$. By LaSalle's Invariance Principle, solutions approach the largest invariant set contained in $\{I = 0\}$, which is the singleton $\{E_0\}$. Hence the disease-free equilibrium is globally asymptotically stable [19].

4 Results and Discussion

The model shows that the population will never go outside a possible region thus all compartments are non-negative. Equilibrium of disease free has no vulnerable and poor individuals. The threshold is given by the basic reproduction number of \mathcal{R}_0 , when $\mathcal{R}_0 < 1$, poverty cannot propagate and the DFE is stable. Eigenvalue analysis proves that when \mathcal{R}_0 is less than 1, all eigenvalues are negative which is verified by local stability. The global asymptotic stability in the feasible region can be seen in the Lyapunov-based analysis. In general, the model summarizes the short-term and long-term dynamics of the spread and recovery of poverty.

5 Conclusion

The transmission and relapse of economic vulnerability can be well understood using a modified SEIR model of poverty dynamics. The analysis establishes that these solutions are all positive and bounded and therefore the model is biologically and socially viable. Equilibrium with no disease happens and its stability is determined by the basic reproduction number, which is \mathcal{R}_0 . The analysis of local stability through eigenvalues and upper-triangular reduction indicates that when \mathcal{R}_0 is less than

one, the DFE is stable. An approach to global stability is achieved with the help of a Lyapunov function, which ensures a terminal convergence. In the future research, it might be possible to include stochastic elements or time-varying interventions to make it more realistic.

References

- [1] Z. Ali, S. H. F. Naqvi, A. R. A. Furkan, Z. A. Kalhor, and M. H. Laghari, “Decomposing multidimensional poverty in Pakistan using Alkire and Foster methodology,” *Pakistan Journal of Social Science Review*, vol. 5, no. 1, pp. 710–723, 2026. [Online]. Available: <http://pjsrjournal.com/index.php/Journal/article/view/530>
- [2] Z. A. Qureshi, M. A. Solangi, A. A. Shaikh, and J. G. Sahito, “Measuring multidimensional parameters of poverty using Alkire and Foster methodology in Qasimabad: A case study,” *Journal of Poverty, Investment and Development*, pp. 112–116, 2018. [Online]. Available: <https://iiste.org/Journals/index.php/JPID/article/view/42894>
- [3] L. Kong, M. Duan, J. Shi, J. Hong, Z. Chang, and Z. Zhang, “Compartmental structures used in modeling COVID-19: A scoping review,” *Infectious Diseases of Poverty*, vol. 11, no. 1, p. 72, 2022. [Online]. Available: <https://link.springer.com/article/10.1186/s40249-022-01001-y>
- [4] H. M. Srivastava, I. Area, and J. J. Nieto, “Power-series solution of compartmental epidemiological models,” *Mathematical Biosciences Engineering*, vol. 18, no. 4, pp. 3274–3290, 2021. [Online]. Available: <https://doi.org/10.3934/mbe.2021163>
- [5] H. W. Hethcote, “The mathematics of infectious diseases,” *SIAM Review*, vol. 42, no. 4, pp. 599–653, 2000. [Online]. Available: <https://doi.org/10.1137/S0036144500371907>
- [6] J. M. Epstein, “Agent-based computational models and generative social science,” *Complexity*, vol. 4, no. 5, pp. 41–60, 1999. [Online]. Available: [https://onlinelibrary.wiley.com/doi/10.1002/\(SICI\)1099-0526\(199905/06\)4:5%3C41::AID-CPLX9%3E3.0.CO;2-F](https://onlinelibrary.wiley.com/doi/10.1002/(SICI)1099-0526(199905/06)4:5%3C41::AID-CPLX9%3E3.0.CO;2-F)
- [7] W. O. Kermack and A. G. McKendrick, “A contribution to the mathematical theory of epidemics,” *Proceedings of the Royal Society A*, vol. 115, no. 772, pp. 700–721, 1927. [Online]. Available: <https://doi.org/10.1098/rspa.1927.0118>

[8] O. Diekmann and J. A. P. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation*. Chichester: John Wiley & Sons, 2000.

[9] R. M. Anderson and R. M. May, *Infectious Diseases of Humans: Dynamics and Control*. Oxford: Oxford University Press, 1991.

[10] C.-W. Chang, Z. A. Qureshi, S. Qureshi, A. A. Shaikh, and M. Y. Shahani, “Real-Data-Based study on divorce dynamics and elimination strategies using nonlinear differential equations,” *Mathematics*, vol. 12, no. 16, p. 2552, 2024. [Online]. Available: <https://doi.org/10.3390/math12162552>

[11] Z. Ali, S. H. F. Naqvi, and A. R. A. Furkan, “Mathematical modeling and analysis of an SIR epidemic model with therapy: Existence, positivity, and stability study,” *Spectrum of Engineering Sciences*, vol. 4, no. 2, pp. 832–842, 2026. [Online]. Available: <https://thesesjournal.com/index.php/1/article/view/2040/1525>

[12] C. Hernandez-Suarez and O. M. Lopez, “A simple and intuitive method to calculate R_0 in complex epidemic models,” *arXiv (Cornell University)*, 2022. [Online]. Available: <https://arxiv.org/abs/2208.11509>

[13] A. Perasso, “An introduction to the basic reproduction number in mathematical epidemiology,” *ESAIM*, vol. 62, pp. 123–138, 2018. [Online]. Available: <https://doi.org/10.1051/proc/201862123>

[14] P. Van Den Driessche and J. Watmough, “Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission,” *Mathematical Biosciences*, vol. 180, no. 1–2, pp. 29–48, 2002. [Online]. Available: [https://doi.org/10.1016/s0025-5564\(02\)00108-6](https://doi.org/10.1016/s0025-5564(02)00108-6)

[15] S. Pathak *et al.*, “Mathematical analysis of an infectious disease model for local stability,” *Nonlinear Dynamics and Systems Theory*, 2025. [Online]. Available: <https://doi.org/10.1007/s44198-025-00333-1>

[16] J. J. He, A. Aljohani, S. Mustafa, A. Shokri, M. M. Khalsaraci, and H. Mukalazi, “Stability analysis of a nonlinear malaria transmission epidemic model using an effective numerical scheme,” *Scientific Reports*, vol. 14, no. 1, p. 17413, 2024. [Online]. Available: <https://doi.org/10.1038/s41598-024-66503-1>

[17] I. Ullah, N. Ali, I. U. Haq, I. Ahmad, M. D. Albalwi, and M. H. A. Biswas, “Analysis of COVID-19 disease model: Backward bifurcation and impact of pharmaceutical and nonpharmaceutical interventions,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2024, pp. 1–17, 2024. [Online]. Available: <https://doi.org/10.1155/2024/6069996>

[18] W. Govaerts and M. Van Daele, “Attractivity of the disease-free subspace in epidemic models,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 152, p. 109260, 2025. [Online]. Available: <https://doi.org/10.1016/j.cnsns.2025.109260>

[19] J. P. La Salle, *The Stability of Dynamical Systems*. SIAM, 1976.